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# THE ASYMPTOTICALLY REGULARITY OF MAPS AND SEQUENCES IN PARTIAL CONE METRIC SPACES WITH APPLICATION

JEROLINA FERNANDEZ[1], KALPANA SAXENA[2], NEERAJ MALVIYA[3]

**ABSTRACT :** In present paper, we define asymptotically regular sequences and maps in partial cone metric space and prove some fixed point theorems for such maps. Our results extend the results of [9] in partial cone metric space.

**Key words :** Partial cone metric space, asymptotically regular sequences and maps, fixed point.
**2010 AMS Subject Classification.** 54H25, 47H10

## 1. INTRODUCTION

In 1980, Rzepecki [15] introduced a generalized metric by replacing the set of real numbers with a Banach space E in the metric function where P is a normal cone in E with partial order $\leq$.

Lin [8] considered the notion of cone metric spaces by replacing real numbers with a cone P in the metric function in which it is called a K-metric. Without mentioning the papers of Lin and Rzepecki, in 2007, Huang and Zhang [5] announced the notion of cone metric spaces (CMS) by replacing real numbers with an ordering Banach space. The authors obtained some fixed point theorems for mappings satisfying different contractive conditions. Afterwards several fixed and common fixed point results on cone metric spaces were introduced in (see [1], [2], [12], [13], [16], [19]).

Recently, in 2013, based on the definition of cone metric spaces and partial metric spaces, Sonmez [17] defined a partial cone metric space. The author developed some fixed point theorems in this generalized setting. Very recently, without using the normality of the cone, Malhotra et al. [10] and Jiang and Li [7] extended the results of [17, 18] to θ-complete partial cone metric spaces.

In the present paper, we define asymptotically regular maps and sequences and present some fixed point results for these maps in partial cone metric space.

## 2. PRELIMINARIES

First, we invite some standard notations and definitions in cone metric spaces and partial cone metric spaces.

A cone $P$ is a subset of a real Banach space $E$ such that

(i) $P$ is closed, nonempty and $P \neq \{0\}$;

(ii) if a, b are nonnegative real numbers and $x, y \in P$, then $ax + by \in P$;

(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subseteq E$, we define a partial ordering $\leq$ with respect to $P$ by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, int $P$ denotes the interior of $P$.

The cone $P$ is called normal if there is a number $k > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\| x \| \leq k \| y \|$.

The least positive number $k$ satisfying above is called the normal constant of $P$.

The cone $P$ is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is sequence such that $x_1 \leq x_2 \leq \dots \leq x_n \leq \dots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\| x_n - x \| \to 0 (n \to \infty)$. Equivalently, the cone $P$ is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

**Definition 2.1.** [5] Let $X$ be a nonempty set. Suppose the mapping $d : X \times X \to P$ satisfies

($d1$) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

($d2$) $d(x, y) = d(y, x)$ for all $x, y \in X$;

($d3$) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $d$ is called a cone metric on $X$, and $(X, d)$ is called a cone metric space.

**Definition 2.2[17]** A partial cone metric on a nonempty set $X$ is a function $p : X \times X \to P$ such that for all $x, y, z \in X$:

$(p1)$ $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

$(p2)$ $\mathbf{0} \leq p(x, x) \leq p(x, y)$,

$(p3)$ $p(x, y) = p(y, x)$,

$(p4)$ $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial cone metric space is a pair $(X, p)$ such that $X$ is a nonempty set and $p$ is a partial cone metric on $X$. It is clear that, if $p(x, y) = \mathbf{0}$, then from $(p1)$ and $(p2)x = y$. But if $x = y$, $p(x, y)$ may not be $\mathbf{0}$.

A cone metric space is a partial cone metric space. But there are partial cone metric spaces which are not cone metric spaces. The following an example illustrate a partial cone metric space but not a cone metric space.

**Example 2.3.** [17] Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = R^+$ and $p : X \times X \to P$ is defined by

$$p(x, y) = (\max \{x, y\}, \alpha \max\{x, y\})$$

where $\alpha \geq 0$ is a constant. Then $(X, p)$ is a partial cone metric space which is not a cone metric space.

**Theorem 2.4.**[17] Any partial cone metric space $(X, p)$ is a topological space.

**Theorem 2.5.**[17] Let $(X, p)$ be a partial cone metric space and P be a normal cone with normal constant K, then $(X, p)$ is $T_0$.

**Definition 2.6.**[17] Let $(X, p)$ be a partial cone metric space. Let $\{x_n\}$ be a sequence in $X$ and $x \in X$. If for every $c \in \text{int}P$, there is N such that for all $n > N$, $p(x_n, x) \ll c + p(x, x)$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to $x$, and $x$ is the limit of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or, $x_n \to x$ ans $n \to \infty$.

**Theorem 2.7.**[17] Let $(X, p)$ be a partial cone metric space, $P$ be a normal cone with

normal constant K. Let $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ converges to $x$ if and only if $p(x_n, x) \to p(x, x)$ as $n \to \infty$.

Sonmez [17] also noted that if $(X, p)$ is a partial cone metric space. $P$ be a normal cone with normal constant $K$ and

$$p(x_n, x) \to p(x, x)\ (n \to \infty), \text{ then } p(x_n, x_n) \to p(x, x) \text{ as } n \to \infty.$$

**Lemma 2.8.**[17] Let $\{x_n\}$ be a sequence in partial cone metric space $(X, p)$. If a point $x$ is the limit of $\{x_n\}$ and $p(y, y) = p(y, x)$ then $y$ is the limit point of $\{x_n\}$.

**Definition 2.9.** [17] Let $(X, p)$ be a partial cone metric space. $\{x_n\}$ be a sequence in $X$. $\{x_n\}$ is Cauchy sequence if there is $a \in P$ such that for every $\varepsilon > 0$ there is N such that for all $n, m > N$

$$\| p(x_n, x_m) - a \| < \varepsilon.$$

**Definition 2.10.**[17] A partial cone metric space $(X, p)$ is said to be complete if every Cauchy sequence in $(X, p)$ is convergent in $(X, p)$.

**Theorem 2.11.**[17] Let $(X, p)$ be a partial cone metric space. If $\{x_n\}$ is a Cauchy sequence in $(X, p)$, then it is a Cauchy sequence in the cone metric space $(X, d)$.

**Proposition 2.12**[3] : Let $P$ be a cone in a real Banach space $E$. If $a \in P$ and $a \leq ka$, for some $k \in [0, 1)$ then $a = \mathbf{0}$.

**Definition 2.13[9]:** Let $(X, p)$ and $(X', p')$ be a partial cone metric space. Then a function $f: X \to X'$ is said to be continuous at a point $x \in X$ if and only if it is sequentially continuous at $x$, that is whenever $\{x_n\}$ is convergent to $x$ we have $\{fx_n\}$ is convergent to $f(x)$.

**Lemma 2.14[9]** : Let $(X, p)$ be a partial cone metric space, $P$ be a normal cone with normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in $X$ and suppose that $x_n \to x$, $y_n \to y$ as $n \to \infty$. Then $p(x_n, y_n) \to p(x, y)$ as $n \to \infty$.

**Proposition 2.15**[6] : Let $(X, d)$ be a cone metric space and $P$ be a cone in a real Banach space $E$. If $u \leq v$, $v \ll w$ then $u \ll w$.

## 3. ASYMPTOTICALLY REGULAR SEQUENCES AND MAPS

Here we will define asymptotically regular sequences and maps in partial cone metric spaces.

**Definition 3.1:** Let $(X, p)$ be a partial cone metric space. A sequence $\{x_n\}$ in $X$ is said to be asymptotically $T$-regular if $\lim_{n\to\infty} p(x_n, Tx_n) = \mathbf{0}$ or $\lim_{n\to\infty} p(Tx_n, x_n) = \mathbf{0}$.

**Example 3.2.** Let $E = R^2$, $P = \{(x, y) \in E : x\, y \geq 0\}$, $X = R^+$ and $p : X \times X \to P$ is defined by

$p(x, y) = [\max\{x, y\}, \alpha \max\{x, y\}] \quad \forall\, x, y \in X$

where $\alpha \geq 0$ is a constant. Then $(X, p)$ is a partial cone metric space. Now let $T$ be a self map of $X$ such that $Tx = \frac{x}{2}$ and choose a sequence $\{x_n\}$, $x_n \neq 0$ for any positive integer $n$, which converges to zero. We deduce that

$$\begin{aligned}
\lim_{n\to\infty} p(x_n, Tx_n) &= \lim_{n\to\infty} \left(\max\{x_n, Tx_n\}, \alpha \max\{x_n, Tx_n\}\right) \\
&= \lim_{n\to\infty} \left(\max\left\{x_n, \tfrac{x_n}{2}\right\}, \alpha \max\left\{x_n, \tfrac{x_n}{2}\right\}\right) \\
&= \lim_{n\to\infty} (x_n, \alpha x_n) \\
&= (0, 0) \\
&= \mathbf{0}.
\end{aligned}$$

Hence $\{x_n\}$ is an asymptotically $T$-regular sequence in $(X, p)$.

**Definition 3.3:** Let $(X, p)$ be a partial cone metric space. A mapping $T$ of $X$ into itself is said to be asymptotically regular at a point $x$ in $X$ if $\lim_{n\to\infty} p\left(T^n x, T^{n+1} x\right) = \mathbf{0}$ or $\lim_{n\to\infty} p\left(T^{n+1} x, T^n x\right) = \mathbf{0}$ where $T^n x$ denotes the $n^{th}$ *iterate of* $T$ at $x$.

**Example 3.4.** Let $(X, p)$ is a partial cone metric space which is defined in example 3.2 and let $T$ be a self map of $X$ such that $Tx = \frac{x}{4}$ where $x \in X$. Then, we have

$$\lim_{n\to\infty} p\left(T^n x, T^{n+1} x\right) = \lim_{n\to\infty} \left(\max\left\{T^n x, T^{n+1} x\right\}, \alpha \max\left\{T^n x, T^{n+1} x\right\}\right)$$

$$= \lim_{n\to\infty}\left(\max\left\{\frac{x}{2^n}, \frac{x}{2^{n+1}}\right\}, \alpha \max\left\{\frac{x}{2^n}, \frac{x}{2^{n+1}}\right\}\right)$$

$$= \lim_{n\to\infty}\left(\frac{x}{2^n}, \alpha\frac{x}{2^n}\right)$$

$$= (0, \alpha 0)$$

$$= 0.$$

Hence $T$ is an asymptotically regular map at all points of $X$.

## 4. MAIN RESULTS

As an application of asymptotically regular maps and sequences, we present some fixed point theorems in the partial cone metric spaces.

**Theorem 4.1.** Let $(X, p)$ be a complete partial cone metric space ant $T$ be a self mapping of $X$ satisfying the inequality.

$$p(Tx, Ty) \le a_1 p(x, Tx) + a_2 p(y, Ty) + a_3 p(x, Ty) + a_4 p(y, Tx) + a_5 p(x, y) \quad \text{...(4.1.1)}$$

$$\text{for all } x, y \in X \text{ where } a_1, a_2, a_3, a_4, a_5 \ge 0 \text{ and } (a_3 + a_4 + a_5) < 1 \quad \text{...(4.1.2)}$$

If there exists and asymptotically $T$-regular sequence in $X$, then $T$ has a unique fixed point.

**Proof :** Let $\{x_n\}$ be an asymptotically $T$-regular sequence in $X$. Then

$$\begin{aligned}
p(x_n, x_m) &\le p(x_n, Tx_n) + p(Tx_n, x_m) - p(Tx_n, Tx_n)\\
&\le p(x_n, Tx_n) + p(x_m, Tx_n) - p(Tx_n, Tx_n)\\
&\le p(x_n, Tx_n) + p(x_m, Tx_m) + p(Tx_m, Tx_n) - p(Tx_m, Tx_m) - p(Tx_n, Tx_n)\\
&\le p(x_n, Tx_n) + p(x_m, Tx_m) + p(Tx_m, Tx_n)\\
&\le p(x_n, Tx_n) + p(x_m, Tx_m) + a_1 p(x_m, Tx_m) + a_2 p(x_n, Tx_n) + a_3 p(x_m, Tx_n)\\
&\qquad + a_4 p(x_n, Tx_m) + a_5 p(x_m, x_n)
\end{aligned}$$

$$\leq p(x_n, Tx_n) + p(x_m, Tx_m) + a_1 p(x_m, Tx_m) + a_2 p(x_n, Tx_n) + a_3[p(x_m, x_n) + p(x_n, Tx_n) - p(x_n, x_n)] + a_4[p(x_n, x_m) + p(Tx_m, x_m) - p(x_m, x_m)] + a_5 p(x_m, x_n)$$

$$\leq p(x_n, Tx_n) + p(x_m, Tx_m) + a_1 p(x_m, Tx_m) + a_2 p(x_n, Tx_n) + a_3[p(x_n, x_m) + p(x_n, Tx_n)] + a_4[p(x_n, x_m) + p(Tx_m, x_m)] + a_5 p(x_m, x_n)$$

$$= (1 + a_2 + a_3)p(x_n, Tx_n) + (1 + a_1 + a_4)p(x_m, Tx_m) + (a_3 + a_4 + a_5)p(x_n, x_m)$$

[by $p_3$]

$$\Rightarrow [1 - (a_3 + a_4 + a_5)]p(x_n, x_m) \leq (1 + a_2 + a_3)p(x_n, Tx_n) + (1 + a_1 + a_4)p(x_m, Tx_m)$$

So, $p(x_n, x_m) \leq \dfrac{(1+a_2+a_3)}{[1-(a_3+a_4+a_5)]}p(x_n, Tx_n) + \dfrac{(1+a_1+a_4)}{[1-(a_3+a_4+a_5)]}p(x_m, Tx_m)$

Since $\{x_n\}$ is an asymptotically $T$-regular sequence and $m > n$. Therefore $p(x_n, Tx_n) \to \mathbf{0}$ and $p(x_m, Tx_m) \to \mathbf{0}$ when $n \to \infty$

This implies that $p(x_n, x_m) \to \mathbf{0}$ as $n, m \to \infty$ . Hence $\{x_n\}$ is a Cauchy sequence. By completeness of $X$, there is $x \in X$ such that $x_n \to x$ as $n \to \infty$. Therefore

$$\lim_{n\to\infty} p(x_n, x) = p(x, x) = \lim_{n\to\infty} p(x_n, x_n) = \mathbf{0}. \qquad ...(4.1.3)$$

**Existence of Fixed Point:**

Consider, $p(Tx, x) \leq p(Tx, Tx_n) + p(Tx_n, x) - p(Tx_n, Tx_n)$

$$\leq p(Tx, Tx_n) + p(Tx_n, x)$$

$$\leq a_1 p(x, Tx) + a_2 p(x, Tx_n) + a_3 p(x, Tx_n) + a_4 p(x_n, Tx) + a_5 p(x, x_n) + p(Tx_n, x)$$

$$\leq a_1 p(x, Tx) + a_2 p(x_n, Tx_n) + a_3[p(x, x_n) + p(x_n, Tx_n) - p(x_n, x_n)] + a_4[p(x_n, x) + p(x, Tx) - p(x, x)] + a_5 p(x, x_n) + [p(Tx_n, x_n) + p(x_n, x) - p(x_n, x_n)]$$

$$\leq a_1 p(x, Tx) + a_2 p(x_n, Tx_n) + a_3[p(x, x_n) + p(x_n, Tx_n)] + a_4[p(x_n, x) + p(x, Tx)] + a_5 p(x, x_n) + [p(Tx_n, x_n) + p(x_n, x)]$$

$$= (a_1 + a_4)p(Tx, x) + (1 + a_2 + a_3)p(Tx_n, x_n) + (1 + a_3 + a_4 + a_5)p(x, x_n)$$

So, $p(Tx, x) \leq \frac{(1+a_2+a_3)}{1-(a_1+a_4)} p(Tx_n, x_n) + \frac{(1+a_3+a_4+a_5)}{1-(a_1+a_4)} p(x, x_n)$

Since $\{x_n\}$ is an asymptotically $T$-regular sequence and $\{x_n\}$ is a Cauchy sequence in $X$. Therefore $x_n \to x$ implies that $p(x_n, Tx_n) \to \mathbf{0}$ and $p(x, x_n) \to \mathbf{0}$ as $n \to \infty$ by (4.1.3). So $\|p(Tx, x)\| = \mathbf{0}$.

$\Rightarrow Tx = x$.

**Uniqueness :** Let $z$ be another fixed point of $T$.

Then $p(x, z) = p(Tx, Tz)$

$\leq a_1 p(x, Tx) + a_2 p(z, Tz) + a_3 p(x, Tz) + a_4 p(z, Tx) + a_5 p(x, z)$

$= a_1 p(x, x) + a_2 p(z, z) + a_3 p(x, z) + a_4 p(z, x) + a_5 p(x, z)$

So, $p(x, z) \leq (a_3 + a_4 + a_5) p(x, z)$

$\Rightarrow p(x, z) = \mathbf{0}$ [by Prop. 2.12 and $(a_3 + a_4 + a_5) < 1$]

$\Rightarrow x = z$

This completes the proof of the theorem 4.1.

**Theorem 4.2.** Let $(X, p)$ be a complete partial cone metric space and $T$ a self mapping of $X$, satisfying the inequality (4.1.1) for all $x, y \in X$ and $a_1, a_2, a_3, a_4, a_5 \geq 0$ and $\max\{(a_1 + a_4), (a_3 + a_4 + a_5)\} < 1$

If $T$ is asymptotically regular at some fixed point $x$ of $X$, then there exists a unique fixed point of $T$.

**Proof :** Let $T$ be an asymptotically reguar at $x_0 \in X$. Consider the sequence $\{T^n x_0\}$ then for all $m, n \geq 1$

$p(T^m x_0, T^n x_0) \leq a_1 p(T^{m-1} x_0, T^m x_0) + a_2 p(T^{n-1} x_0, T^n x_0) + a_3 p(T^{m-1} x_0, T^n x_0)$

$+ a_4 p(T^{n-1} x_0, T^m x_0) + a_5 p(T^{m-1} x_0, T^{n-1} x_0)$

$$\leq a_1p(T^{m-1}x_0,\ T^mx_0) + a_2p(T^{n-1}x_0,\ T^nx_0) + a_3[p(T^{m-1}x_0,\ T^mx_0)$$
$$+ p(T^mx_0,\ T^nx_0) - p(T^mx_0,\ T^mx_0)] + a_4[p(T^{n-1}x_0,\ T^nx_0) + p(T^mx_0,\ T^nx_0)$$
$$- p(T^nx_0,\ T^nx_0)] + a_5[p(T^{m-1}x_0,\ T^mx_0) + p(T^{n-1}x_0,\ T^mx_0) - p(T^mx_0,\ T^mx_0)]$$
$$\leq a_1p(T^{m-1}x_0, T^mx_0) + a_2p(T^{n-1}x_0, T^nx_0) + a_3[p(T^{m-1}x_0, T^mx_0) + p(T^mx_0, T^nx_0)]$$
$$+ a_4[p(T^{n-1}x_0, T^nx_0) + p(T^mx_0, T^nx_0)] + a_5[p(T^{m-1}x_0, T^mx_0) + p(T^{n-1}x_0, T^nx_0)$$
$$+ p(T^mx_0,\ T^nx_0) - p(T^mx_0,\ T^mx_0) - p(T^nx_0,\ T^nx_0)]$$
$$\leq a_1p(T^{m-1}x_0, T^mx_0) + a_2p(T^{n-1}x_0, T^nx_0) + a_3[p(T^{m-1}x_0, T^mx_0) + p(T^mx_0, T^nx_0)]$$
$$+ a_4[p(T^{n-1}x_0, T^nx_0) + p(T^mx_0, T^nx_0)] + a_5[p(T^{m-1}x_0, T^mx_0) + p(T^{n-1}x_0, T^nx_0)$$
$$+ p(T^mx_0,\ T^nx_0)]$$
$$= (a_1 + a_3 + a_5)p(T^{m-1}x_0,\ T^mx_0) + (a_2 + a_4 + a_5)p(T^{n-1}x_0,\ T^nx_0)$$
$$+ (a_3 + a_4 + a_5)p(T^mx_0,\ T^nx_0)$$

So, $p(T^mx_0,\ T^nx_0) \leq \dfrac{(a_1+a_3+a_5)}{1-(a_3+a_4+a_5)}p(T^{m-1}x_0,\ T^mx_0) + \dfrac{(a_2+a_4+a_5)}{1-(a_3+a_4+a_5)}p(T^{n-1}x_0,\ T^nx_0)$

Since $T$ is an asymptotically regular at $x_0$, therefore $p(T^{m-1}x_0,\ T^mx_0) \to \mathbf{0}$ and $p(T^{n-1}x_0,\ T^nx_0)$ $\to \mathbf{0}$ as $m,\ n \to \infty$.

This implies that $p(T^mx_0,\ T^nx_0) \to 0$ as $n,\ m \to \infty$. Hence $\{T^nx_0\}$ is a Cauchy sequence in $X$. By completeness of $X$, there is $x \in X$ such that $T^nx_0 \to x$ as $n \to \infty$. Therefore

$$\lim_{n\to\infty} p\left(T^nx_0,\ x\right) = p(x,\ x) = \lim_{n\to\infty} p\left(T^nx_0,\ T^nx_0\right) = \mathbf{0}.$$

Therefore, $\{T^nx_0\}$ is a Cauchy sequence in $X$ which is complete space. So, $\{T^nx_0\} \to x \in X$.

Now, we claim that $x$ is a fixed point to $T$. For this we have,

$$p(Tx,\ x) \leq p(Tx\ T^nx_0) + p(T^nx_0,\ x) - p(T^nx_0,\ T^nx_0)$$
$$\leq a_1p(x,\ Tx) + a_2p(T^{n-1}x_0,\ T^nx_0) + a_3p(x,\ T^nx_0) + a_4p(T^{n-1}x_0,\ Tx)$$
$$+ a_5p(x,\ T^{n-1}x_0) + p(T^nx_0,\ x)$$

$$\leq a_1 p(x, Tx) + a_2 p(T^{n-1}x_0, T^n x_0) + a_3 p(x, T^n x_0) + a_4[p(T^{n-1}x_0, T^n x_0) + p(T^n x_0, Tx)$$
$$- p(T^n x_0, T^n x_0)] + a_5[p(x, T^n x_0) + p(T^{n-1}x_0, T^n x_0) - p(T^n x_0, T^n x_0)] + p(T^n x_0, x)$$
$$\leq a_1 p(x, Tx) + a_2 p(T^{n-1}x_0, T^n x_0) + a_3 p(x, T^n x_0) + a_4[p(T^{n-1}x_0, T^n x_0) + p(T^n x_0, Tx)]$$
$$+ a_5[p(x, T^n x_0) + p(T^{n-1}x_0, T^n x_0)] + p(T^n x_0, x)$$

$p(Tx, x) \leq (a_1 + a_4) p(x, Tx)$ [as $n \to \infty$] [Since $\{T^{n-1}x_0\}$ is a subsequence of $\{T^n x_0\}$]

$\Rightarrow p(Tx, x) = 0$ [by Prop 2.12 and as $a_1 + a_4 < 1$]

$\Rightarrow Tx = x$

The uniqueness of the fixed point $x$ follows as in theorem 4.1 using $(a_3 + a_4 + a_5) < 1$. This completes the proof of the theorem 4.2.

The following example demonstrates theorem 4.2.

**Example 4.3 :** Let $(X, p)$ is a partial cone metric space which is defined in example 3.2 and let $T$ be a self map of $X$ such that $Tx = \frac{x}{2}$ where $x \in X$. Clearly $T$ is an asymptotically regular map at all points of $X$. If we take $a_1 = a_2 = a_3 = a_4 = 0$ and $a_5 = \frac{1}{2}$. Then the contractive condition (4.1.1) holds trivially good and 0 is the unique fixed point of the map $T$.

**Conclusion :** The asymptotically regularity of the mapping $T$ satisfies the Hardy Rogers contraction condition. It is actually a consequence of $\sum_i^5 a_i < 1$. Thus the theorem 4.1 and the theorem 4.2 extend results due to Hardy Rogers [4] in partial cone metric spaces. It is also worth mentioning that our condition on controls constants says that $\sum_{i=1}^5 a_i$ may exceed 1.

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# SOME CURVATURE PROPERTIES OF *LP*-SASAKIAN MANIFOLDS

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**ABSTRACT :** The object of the present paper is to study some curvature conditions in *LP*-Sasakian manifolds.

**Key words and Phrases :** *LP*-Sasakian manifolds, *M*-projective curvature tensor, Pseudo projective curvature tensor, Conformal curvature tensor, Quasi conformal curvature tensor.

**Mathematics Subject Classifications (2010).** 53C15, 53C25

## 1. INTRODUCTION

The notion of Lorentzian Para-Sasakian manifolds was introduced by K. Matsumoto [1] in 1989. Then Mihai and Rosca [2] introduced the same notion independently and obtained several results on this manifold. *LP*-Sasakian manifolds have also been studied by Matsumoto and Mihai [3], Mihai et al. [9], Venkatesha and Bagewadi [4], and many others.

On the other hand, Pokhariyal and Mishra [5] have introduced new curvature tensor called a *M*-projective curvature tensor in a Riemannian manifold and studied its properties. Further, Pokhariyal [6] has studied some properties of this curvature tensor in a Sasakian manifold. Chaubey and Ojha [7], Singh et al. [11] and may others geometers have studied this curvature tensor.

In the present paper we study some curvature conditions on *LP*-Sasakian manifolds. The paper is organized as follows : Section 2 consists the basic definitions of Einstein and $\eta$-Einstein manifold. Section 3 is about the study of *M*-projective curvature tensor in *LP*-Sasakian manifolds. Section 4 is devoted to the study of an *LP*-Sasakian manifold satisfying $P(\xi, X) \cdot W^* = 0$ and $W^*(\xi, X) \cdot P = 0$. Section 5 deals with properties of conformal curvature tensor satisfying $C(\xi, X) \cdot W^* = 0$ and $W^*(\xi, X) \cdot C = 0$. Finally, we consider *LP*- Sasakian maniolds satisfying $\overline{C}(\xi, X) \cdot W^* = 0$.

## 2. PRELIMINARIES

An $n$-dimensional differentiable manifold $M^n$ is called a Lorentzian Para-Sasakian (briefly $LP$-Sasakian) manifolds ([1], [2]) if it admits a (1, 1) tensor field $\phi$, a contravariant vector field $\xi$, a covariant vector field $\eta$ and the Lorentzian metric $g$, which satisfy

$$\phi^2 X = X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = -1, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X), \tag{2.4}$$

$$(\nabla_X \phi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.5}$$

$$\nabla_X \xi = \phi X, \tag{2.6}$$

for any vector fields $X$ and $Y$, where $\nabla$ denotes the operator of covariant differentiation with respect to the Lorentzian metric $g$. It can be easily seen that in an $LP$-Sasakian manifold, the following relations hold :

$$\phi(\xi) = 0 \quad \eta(\phi X) = 0, \quad rank(\phi) = (n - 1) \tag{2.7}$$

If we put

$$\Phi(X, Y) = g(\phi X, Y), \tag{2.8}$$

for any vectort fields $X$ and $Y$, then the tensor field $\Phi(X, Y)$ is symmetric (0, 2) tensor field [1]. Also, since the 1 - form $\eta$ is closed in an $LP$ - Sasakian manifold, we have ([1], [10])

$$(\nabla_X \eta)(Y) = \phi(X, Y), \quad \phi(X, \xi) = 0, \tag{2.9}$$

for any vector fields $X$ and $Y$.

Let $M^n$ be an $n$-dimensional $LP$-Sasakian manifold with structure $(\phi, \xi, \eta, g)$ then we have ([3], [10])

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{2.10}$$

$$R(\xi, X)Y = -R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.11}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.12}$$

$$S(X, \xi) = (n-1)\eta(X), \tag{2.13}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{2.14}$$

for any vector fields $X$, $Y$, $Z$, where $R$ is the Riemannian curvature tensor.

An $LP$-Sasakian manifold $M^n (n > 2)$ is said to be Einstein manifold if its Ricci tensor $S$ is of the form

$$S(X, Y) = kg(X, Y), \tag{2.15}$$

where $k$ is constant.

An $LP$-Sasakian manifold $M^n$ is said to be an $\eta$-Einstein manifold if its Ricci tensor $S$ is of the form

$$S(X, Y) = \alpha g(X, Y) + \beta\eta(X)\eta(Y), \tag{2.16}$$

for arbitrary vector fields $X$ and $Y$, where $\alpha$ and $\beta$ are smooth functions.

## 3. M-PROJECTIVE CURVATURE TENSOR OF *LP*-SASAKIAN MANIFOLDS

In 1971, Pokhariyal and Mishra [5] defined a tensor field $W^*$ on a Riemannian manifold $M^n$ as

$$\begin{aligned} W^*(X, Y)Z &= R(X, Y)Z - \frac{1}{2(n-1)}[S(Y, Z)X \\ &\quad - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY], \end{aligned} \tag{3.1}$$

for vector fields $X$, $Y$ and $Z$, where $S$ is the Ricci tensor of type (0, 2) and $Q$ is the Ricci operator.

Putting $X = \xi$ in equation (3.1) and using equatings (2.2), (2.4), (2.11) and (2.13) we get

$$\begin{aligned} W^*(\xi, Y)Z &= -W^*(Y, \xi)Z = \frac{1}{2}[g(Y, Z)\xi - \eta(Z)Y] \\ &\quad - \frac{1}{2(n-1)}[S(Y, Z)\xi - \eta(Z)QY]. \end{aligned} \tag{3.2}$$

Again, putting $Z = \xi$ in the equation (3.1) and using equations (2.4), (2.12) and (2.13), we get

$$W^*(X, Y)\xi = \frac{1}{2}[\eta(Y)X - \eta(X)Y]$$

$$- \frac{1}{2(n-1)}[\eta(Y)QX - \eta(X)QY]. \tag{3.3}$$

Now, taking the inner product of equations (3.1), (3.2) and (3.3) with $\xi$ and using equations (2.2), (2.4) and (2.13), we get

$$\eta(W^*(X, Y)Z) = \frac{1}{2}[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]$$

$$- \frac{1}{2(n-1)}[S(Y, Z)\eta(X) - S(X, Z)\eta(Y)], \tag{3.4}$$

$$\eta(W^*(\xi, Y)Z) = -\eta(W^*(Y, \xi)Z)$$

$$= -\frac{1}{2}g(Y, Z) + \frac{1}{2(n-1)}S(Y, Z), \tag{3.5}$$

and

$$\eta(W^*(X, Y)\xi) = 0, \tag{3.6}$$

respectively.

**Theorem 3.1.** *An LP - Sasakian manifold $M^n$ satisfying the condition $R(\xi, X) \cdot W^* = 0$ is an Einstein manifold.*

**Proof.** Let $R(\xi, X) \cdot W^*(Y, Z)U = 0$. Then we have

$$R(\xi, X)W^*(Y, Z)U - W^*(R(\xi, X)Y, Z)U$$

$$- W^*(Y, R(\xi, X)Z)U - W^*(Y, Z)R(\xi, X)U = 0, \tag{3.7}$$

which on using the equation (2.11), gives

$$g(X, W^*(Y, Z)U)\xi - \eta(W^*(Y, Z)U)X - g(X, Y)W^*(\xi, Z)U$$

$$- g(X, Z)W^*(Y, \xi)U - g(X, U)W^*(Y, Z)\xi + \eta(Y)W^*(X, Z)U$$

$$+ \eta(Z)W^*(Y, X)U + \eta(U)W^*(Y, Z)X. \tag{3.8}$$

Now, taking the inner product of the above equation with $\xi$ and using equations (2.2), (2.4), (2.11), (3.1), (3.4), (3.5) and (3.6), we obtain

$$\begin{aligned}'R(Y, Z, U, X) &= g(X, Y)g(Z, U) - g(X, Z)g(Y, U) \\ &+ \tfrac{1}{2}[g(X, Z)\eta(Y)\eta(U) - g(X, Y)\eta(Z)\eta(U)] \\ &+ \frac{1}{2(n-1)}[S(X, Y)\eta(Z)\eta(U) \\ &- S(X, Z)\eta(Y)\eta(U)]. \end{aligned} \tag{3.9}$$

Taking a frame field and contraction over $Z$ and $U$, we get

$$S(X, Y) = (n - 1)g(X, Y)$$

This shows that $M^n$ is an Einstein manifold.

**Theorem 3.2.** *If an LP-Sasakian manifold $M^n$ satisfies the condition $W^*(\xi, X). R = 0$, then*

$$S(QX, Y) = (n - 1)^2\, g(X, Y).$$

**Proof.** Let $W^*(\xi, X) \cdot R(Y, Z)U = 0$. Then, we have

$$\begin{aligned} &W^*(\xi, X)R(Y, Z)U - R(W^*(\xi, X)Y, Z)U \\ &- R(Y, W^*(\xi, X)Z)U - R(Y, Z)W^*(\xi, X)U = 0, \end{aligned} \tag{3.10}$$

which on using the equation (3.2), gives

$$\begin{aligned} &g(X, R(Y, Z)U)\xi - \eta(R(Y, Z)U)X - g(X, Y)R(\xi, Z)U \\ &+ \eta(Y)R(X, Z)U - g(X, Z)R(Y, \xi)U + \eta(Z)R(Y, X)U \\ &- g(X, U)R(Y, Z)\xi + \eta(U)R(Y, Z)X - \frac{1}{n-1}[S(X, R(Y, Z)U)\xi \\ &- \eta(R(Y, Z)U)QX - S(X, Y)R(\xi, Z)U + \eta(Y)R(QX, Z)U \\ &- S(X, Z)R(Y, \xi)U + \eta(Z)R(Y, QX)U - S(X, U)R(Y, Z)\xi \\ &+ \eta(U)R(Y, Z)QX] = 0. \end{aligned} \tag{3.11}$$

Now, taking the inner product of the above equation with $\xi$ and using equations (2.2), (2.4), (2.11), (2.12) and (2.13), we obtain

$$\begin{aligned}
& g(X, R(Y, Z)U) - g(X, Y)\eta(R(\xi, Z)U) + \eta(Y)\eta(R(X, Z)U) \\
& - g(X, Z)\eta(R(Y, \xi)U) + \eta(Z)\eta(R(Y, X)U) - g(X, U)\eta(R(Y, Z)\xi) \\
& + \eta(U)\eta(R(Y, Z)X) - \frac{1}{n-1}['R(Y, Z, U, QX) - S(X, Y)\eta(R(\xi, Z)U) \\
& + \eta(Y)\eta(R(QX, Z)U) - S(X, Z)\eta(R(Y, \xi)U) + \eta(Z)\eta(R(Y, QX)U) \\
& - S(X, U)\eta(R(Y, Z)\xi) + \eta(U)\eta(R(Y, Z)QX)] = 0. \qquad (3.12)
\end{aligned}$$

Taking a frame field nad contraction over $Z$ and $U$, we get

$$S(QX, Y) = (n-1)^2 g(X, Y).$$

This completes the proof.

**Theorem 3.3.** *If an LP-Sasakian manifold $M^n$ satisfies the condition $W^*(\xi, X). S = 0$, then*

$$S(QX, Y) = -(n-1)^2 g(X, Y) + 2(n-1)S(X, Y).$$

**Proof.** Let $W^*(\xi, X) \cdot S(Y, Z) = 0$. Then, we have

$$S(W^*(\xi, X)Y, Z) + S(Y, W^*(\xi, X)Z) = 0, \qquad (3.13)$$

which on using the equation (3.2), gives

$$\begin{aligned}
& (n-1)[g(X, Y)\eta(Z) + g(X, Z)\eta(Y)] - S(X, Z)\eta(Y) - S(X, Y)\eta(Z) \\
& + \frac{1}{(n-1)}[S(QX, Y)\eta(Z) - S(QX, Z)\eta(Y)] = 0. \qquad (3.14)
\end{aligned}$$

Now, putting $Z = \xi$ in the above equation and using equations (2.2), (2.4) and (2.13), we get

$$S(QX, Y) = -(n-1)^2 g(X, Y) + 2(n-1)S(X, Y).$$

This completes the proof.

## 4. *LP*-SASAKIAN MANIFOLDS SATISFYING $P(\xi, X) \cdot W^* = 0$ AND $W^*(\xi, X) \cdot P = 0$

Projective curvature tensor $P$ of the manifold $M^n$ is given by [8]

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}[S(Y, Z)X - S(X, Z)Y]. \tag{4.1}$$

Putting $X = \xi$ in the above equation and the using equations (2.11) and (2.13), we get

$$P(\xi, Y)Z = -P(Y, \xi)Z = g(Y, Z)\xi - \frac{1}{n-1}S(Y, Z)\xi. \tag{4.2}$$

Again, putting $Z = \xi$ in the equation (4.1) and using the equations (2.12) and (2.13), we get

$$P(X, Y)\xi = 0. \tag{4.3}$$

Now, taking the inner product of equations (4.1), (4.2) and (4.3) with $\xi$, we get

$$\eta(P(X, Y)Z = g(Y, Z)\eta(X) - g(X, Z)\eta(Y)$$

$$- \frac{1}{n-1}[S(Y, Z)\eta(X) - S(X, Z)\eta(Y)], \tag{4.4}$$

$$\eta(P(\xi, Y)Z) = -\eta(P(Y, \xi)Z) = -g(Y, Z) + \frac{1}{n-1}S(Y, Z), \tag{4.5}$$

and

$$\eta(P(X, Y)\xi = 0 \tag{4.6}$$

respectively.

**Theorem 4.1.** *If an LP-Sasakian manifold $M^n$ satisfies the condition $P(\xi, X). W^* = 0$ then*

$$S(QX, Y) = 2(n-1)[S(X, Y) - (n-1)g(X, Y)].$$

**Proof.** Let $P(\xi, X) \cdot W^*(Y, Z)U = 0$. Then, we have

$$P(\xi, X)W^*(Y, Z)U - W^*(P(\xi, X)Y, Z)U$$

$$- W^*(Y, P(\xi, X)Z)U - W^*(Y, Z)P(\xi, X)U = 0. \tag{4.7}$$

which on using the equation (4.2), gives

$$g(X, W^*(Y, Z)U)\xi - g(X, Y)W^*(\xi, Z)U - g(X, Z)W^*(Y, \xi)U$$

$$- g(X, U)W^*(Y, Z)\xi - \frac{1}{n-1}[S(X, W^*(Y, Z)U)\xi - S(X, Y)W^*(\xi, Z)U$$

$$- S(X, Z)W^*(Y, \xi)U - S(X, U)W^*(Y, Z)\xi] = 0. \tag{4.8}$$

Now, taking the inner product of above equation with $\xi$ and using equation (2.2), (2.4), (3.1), (3.4), (3.5) and (3.6), we obtain

$$\frac{1}{(n-1)}{}'R(Y, Z, U, QX) = {}'R(Y, Z, U, X) + \frac{1}{2}[g(X, Z)g(Y, U)$$

$$- g(X, Y)g(Z, U)] - \frac{1}{2(n-1)^2}[g(Z, U)X(QX, Y)$$

$$- g(Y, U)S(QX, Z)]. \tag{4.9}$$

Taking a frame field and contraction over $Z$ and $U$, we get

$$S(QX, Y) = 2(n-1)[S(X, Y) - (n-1)g(X, Y)].$$

This completes the proof.

**Theorem 4.2.** *If a LP-Sasakian manifold $M^n$ satisfies the condition $W^*(\xi, X). P = 0$ then*

$$S(QX, Y) = \frac{n(n-1)^2}{n-2}g(X, Y) + \frac{2n(n-1)}{n-2}S(X, Y)$$

**Proof.** Let $W^*(\xi, X) \cdot P(Y, Z)U = 0$. Then, we have

$$W^*(\xi, X)P(Y, Z)U - P(W^*(\xi, X)Y, Z)U$$

$$- P(Y, W^*(\xi, X)Z)U - P(Y, Z)W^*(\xi, X)U = 0. \tag{4.10}$$

which on using the equation (3.2), gives

$$g(X, P(Y, Z)U)\xi - \eta(P(Y, Z)U)X - g(X, Y)P(\xi, Z)U + \eta(Y)P(X, Z)U$$

$$- g(X, Z)P(Y, \xi)U + \eta(Z)P(Y, X)U - g(X, U)P(Y, Z)\xi + \eta(U)P(Y, Z)X$$

$$- \frac{1}{n-1}[S(X, P(Y, Z)U)\xi - \eta(P(Y, Z)U)QX - S(X, Y)P(\xi, Z)U$$

$$+ \eta(Y)P(QX, Z)U - S(X, Z)P(Y, \xi)U + \eta(Z)P(Y, QX)U$$

$$- S(X, U)P(Y, Z)\xi + \eta(U)P(Y, Z)QX] = 0. \tag{4.11}$$

Now, takiing the inner product of above equation with $\xi$ and using equations (2.2), (2.4), (4.1), (4.4), (4.5) and (4.6), we obtain

$$- {}'R(Y, Z, U, X) + g(X, Y)g(Z, U) - g(X, Z)g(Y, U) + g(X, Z)\eta(Y)\eta(U)$$

$$- g(X, Y)\eta(Z)\eta(U) - \frac{1}{(n-1)}[{}'R(Y, Z, U, QX) + 2S(X, Y)\eta(Z)\eta(U)$$

$$- 2S(X, Z)\eta(Y)\eta(U) + S(X, Z)g(Y, U) - S(X, Y)g(Z, U)]$$

$$+ \frac{1}{(n-1)^2}[S(QX, Z)\eta(Y)\eta(U) - S(QX, Y)\eta(Z)\eta(U)] = 0. \tag{4.12}$$

Taking a frame field and contraction over $Z$ and $U$, we get

$$S(QX, Y) = -\frac{n(n-1)^2}{n-2}g(X, Y) + \frac{2n(n-1)}{n-2}S(X, Y).$$

This completes the proof.

## 5. *LP*-SASAKIAN MANIFOLDS SATISFYING $C(\xi, X)\cdot W^* = 0$ AND $W^*(\xi, X)\cdot C = 0$

Conformal curvature tensor $C$ of the manifold $M^n$ is given by [10]

$$C(X, Y)Z = R(X, Y)Z - \frac{1}{(n-2)}[S(Y, Z)X - S(X, Z)Y$$

$$+ g(Y, Z)QX - g(X, Z)QY]$$

$$+ \frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y]. \tag{5.1}$$

Putting $X = \xi$ in the above equatin and using the equations (2.11) and (2.13), we get

$$C(\xi, Y)Z = -C(Y, \xi)Z = \frac{1+r-n}{(n-1)(n-2)}[g(Y, Z)\xi - \eta(Z)Y]$$

$$- \frac{1}{n-2}[S(Y, Z)\xi - \eta(Z)QY]. \tag{5.2}$$

Again, putting $Z = \xi$ in the equation (5.1) and using the equations (2.12) and (2.13), we get

$$C(X, Y)\xi = \frac{1+r-n}{(n-1)(n-2)}[\eta(Y)X - \eta(X)Y]$$

$$- \frac{1}{n-2}[\eta(Y)QX - \eta(X)QY]. \tag{5.3}$$

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Now, taking the inner product of the equations (5.1) and (5.2) with $\xi$, we get

$$\eta(C(X, Y)Z = \frac{1+r-n}{(n-1)(n-2)}[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]$$

$$- \frac{1}{n-2}[S(Y, Z)\eta(X) - S(X, Z)\eta(Y)], \tag{5.4}$$

$$\eta(C(\xi, Y)Z) = - \eta(C(Y, \xi)Z)$$

$$= \frac{1+r-n}{(n-1)(n-2)}[-g(Y, Z) - \eta(Y)\eta(Z)]$$

$$- \frac{1}{n-2}[-S(Y, Z) - \eta(Z)\eta(QY)]. \tag{5.5}$$

**Theorem 5.1.** *If an LP-Sasakian manifold $M^n$ satisfies the condition $C(\xi, X)$. $W^* = 0$ then*

$$S(QX, Y) = \left(\frac{(n^2-3n+r+2)}{(n-1)}\right)S(X, Y) + (1 + r - n)g(X, Y)$$

**Proof.** Let $C(\xi, X)\cdot W^*(Y, Z)U = 0$. Then, we have

$$C(\xi, X)W^*(Y, Z)U - W^*(C\xi, X)Y, Z)U$$

$$- W^*(Y, C(\xi, X)Z)U - W^*(Y, Z)C(\xi, X)U = 0, \tag{5.6}$$

which on using the equation (5.2), gives

$$\frac{1+r-n}{(n-1)(n-2)}[g(X, W^*(Y, Z)U)\xi - \eta(W^*(Y, Z)U)X$$

$$- g(X, Y)W^*(\xi, Z)U + \eta(Y)W^*(X, Z)U - g(X, Z)W^*(Y, \xi)U$$

$$+ \eta(Z)W^*(Y, X)U - g(X, U)W^*(Y, Z)\xi + \eta(U)W^*(Y, Z)X]$$

$$- \frac{1}{n-2}[S(X, W^*(Y, Z)U)\xi - \eta(W^*(Y, Z)U)QX - S(X, Y)W^*(\xi, Z)U$$

$$+ \eta(Y)W^*(QX, Z)U - S(X, Z)W^*(Y, \xi)U + \eta(Z)W^*(Y, QX)U$$

$$- S(X, U)W^*(Y, Z)\xi + \eta(U)W^*(Y, Z)QX] = 0. \tag{5.7}$$

Now, taking the inner product of the above equation with $\xi$ and using the equations (2.2), (2.4), (3.1), (3.4), (3.5) and (3.6), we obtain

$$\frac{1+r-n}{(n-1)(n-2)}[-'R(Y, Z, U, X) + \frac{1}{2(n-1)}\{g(Z, U)S(X, Y) - g(Y, U)S(X, Z)\}$$

$$+ \frac{1}{2}\{g(X, Y)g(Z, U) - g(X, Z)g(Y, U) + g(X, Z)\eta(Y)\eta(U) - g(X, Y)\eta(Z)\eta(U)\}$$

$$+ \frac{1}{2(n-1)}\{S(X, Y)\eta(Z)\eta(U) - S(Z, X)\eta(Y)\eta(U)\}]$$

$$- \frac{1}{(n-2)}[-'R(Y, Z, U, QX) + \frac{1}{2(n-1)}\{g(Z, U)S(QX, Y) - g(Y, U)S(QX, Z)\}$$

$$+ \frac{1}{2}\{S(X, Y)g(Z, U) - S(X, Z)g(Y, U) + S(X, Z)\eta(Y)\eta(U) - S(X, Y)\eta(Z)\eta(U)\}$$

$$+ \frac{1}{2(n-1)}\{S(QX, Y)\eta(Z)\eta(U) - S(QX, Z)\eta(Y)\eta(U)\}] = 0. \tag{5.8}$$

Taking a frame field and contraction over $Z$ and $U$, we get

$$S(QX, Y) = \left(\frac{(n^2-3n+r+2)}{(n-1)}\right)S(X, Y) + (1 - n + r)g(X, Y).$$

This completes the proof.

**Theorem 5.2.** *If an LP-Sasakian manifold $M^n$ satisfies the condition $W^*(\xi, X)$. $C = 0$, then the manifold is an Einstein manifold.*

**Proof.** Let $W^*(\xi, X) \cdot C(Y, Z)U = 0$. Then, we have

$$W^*(\xi, X)C(Y, Z)U - C(W^*(\xi, X)Y, Z)U$$

$$- C(Y, W^*(\xi, X)Z)U - C(Y, Z)W^*(\xi, X)U = 0, \tag{5.9}$$

which on using the equation (3.2), gives

$$g(X, C(Y, Z)U) - \eta(C(Y, Z)U)X - g(X, Y)C(\xi, Z)U$$

$$+ \eta(Y)C(X, Z)U - g(X, Z)C(Y, \xi)U + \eta(Z)C(Y, X)U$$

$$- g(X, U)C(Y, Z)\xi + \eta(U)C(Y, Z)X - \frac{1}{n-1}[S(X, C(Y, Z)U)\xi$$

$$- \eta(C(Y, Z)U)QX - S(X, Y)C(\xi, Z)U + \eta(Y)C(QX, Z)U$$

$$- S(X, Z)C(Y, \xi)U + \eta(Z)C(Y, QX)U - S(X, U)C(Y, Z)\xi$$

$$+ \eta(U)C(Y, Z)QX] = 0. \tag{5.10}$$

Now, taking the inner product of above equation with $\xi$ and using equations (2.2), (2.4), (5.1), (5.4) and (5.5), we obtain

$$- \frac{1}{(n-1)}{}'R(Y, Z, U, QX) = - {}'R(Y, Z, U, X) + \frac{1}{(n-2)}[S(X, Y)g(Z, U)$$

$$- S(X, Z)g(Y, U) + (n-1)\{g(X, Z)\eta(Y)\eta(U) - g(X, Y)\eta(Z)\eta(U)\}$$

$$+ 2\{S(X, Y)\eta(Z)\eta(U) - S(X, Z)\eta(Y)\eta(U)\} + \frac{1}{(n-1)}\{S(QX, Z)g(Y, U)$$

$$- S(QX, Y)g(Z, U) + S(QX, Z)\eta(Y)\eta(U) - S(QX, Y)\eta(Z)\eta(U)\}]$$

$$- \frac{r}{(n-1)(n-2)}[g(Z, U)g(X, Y) - g(Y, U)g(X, Z) + \frac{1}{(n-1)}\{g(Z, U)S(X, Y)$$

$$- g(Y, U)S(X, Z)\}] + \frac{1+r-n}{(n-1)(n-2)}[g(Z, U)g(X, Y) - g(Y, U)g(X, Z)$$

$$+ \frac{1}{(n-1)}\{S(X, Z)g(Y, U) - S(X, Y)g(Z, U)\}]. \tag{5.11}$$

Taking a frame field and contraction over $Z$ and $U$, we get

$$S(X, Y) = 2rg(X, Y).$$

This completes the proof.

## 6. *LP*-SASAKIAN MANIFOLDS SATISFYING $\overline{C}(\xi, X) \cdot W^* = 0$

The notion of the quasi-conformal curvature tensor $\overline{C}$ was introduced by Yano and Sawaki [10]. They defined the quasi-conformal curvature tensor by

$$\overline{C}(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y$$
$$+ g(Y, Z)QX - g(X, Z)QY]$$
$$- \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(Y, Z)X - g(X, Z)Y], \tag{6.1}$$

where $a$ and $b$ are constants such that $ab \neq 0$. If $a = 1$ and $b = \frac{1}{n-2}$, then above equation reduces to conformal curvature tensor given by (5.1). Thus the conformal curvature tensor $C$ is a particular case of the Quasi-conformal curvature tensor $\overline{C}$.

Putting $X = \xi$ in equation (6.1) and using the equations (2.11) and (2.13), we get

$$\overline{C}(\xi, Y) = -\overline{C}(Y, \xi)Z) = \left[a + b(n-1) - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\right]$$
$$[g(Y, Z)\xi - \eta(Z)Y] + b[S(Y, Z)\xi - \eta(Z)QY]. \tag{6.2}$$

Again, putting $Z = \xi$ in the equation (6.1) and using the equations (2.12) and (2.13), we get

$$\overline{C}(X, Y)\xi = \left[a + b(n-1) - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\right][\eta(Y)X - \eta(X)Y]$$
$$+ b[\eta(Y)QX - \eta(X)QY]. \tag{6.3}$$

Now, taking the inner product of equations (6.1), (6.2) and (6.3) with $\xi$, we get

$$\eta(\overline{C}(X, Y)Z) = \left[a + b(n-1) - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\right][g(Y, Z)\eta(X)$$
$$- g(X, Z)\eta(Y)] + b[\eta(Y)QX - \eta(X)QY], \tag{6.4}$$

$$\eta(\overline{C}(\xi, Y)Z) = -\eta(C(Y, \xi)Z)$$
$$= \left[a + b(n-1) - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\right][-g(Y, Z) - \eta(Y, Z) - \eta(Y)\eta(Z)]$$
$$+ b[-S(Y, Z) - \eta(Z)\eta(QY)] \tag{6.5}$$

and

$$\eta(\overline{C}(X, Y)\xi) = 0 \tag{6.6}$$

respectively.

**Theorem 6.1.** *If an LP-Sasakian manifold $M^n$ satisfies the condition $\overline{C}(\xi, X)\cdot W^* = 0$ then*

$$S(QX, Y) = \left[(n-1) - \frac{A}{b}\right]S(X, Y) - \left[\frac{2(n-1)+r}{n}\right]\frac{A}{b}\eta(X)\eta(Y)$$

$$+ \left[\frac{n(n-1)-r}{n}\right]\frac{A}{b}g(X, Y).$$

*where* $A = \left[a + b(n-1) - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\right].$

**Proof.** Let $\overline{C}(\xi, X)\cdot W^*(Y, Z)U = 0$. then, we have

$$\overline{C}(\xi, X)W^*(Y, Z)U - W^*(\overline{C}(\xi, X)Y, Z)U$$

$$- W^*(Y, \overline{C}(\xi, X)Z)U - W^*(Y, Z)\overline{C}(\xi, X)U = 0, \tag{6.7}$$

which on using the equation (6.2), gives

$$A[g(X, W^*(Y, Z)U)\xi - \eta(W^*(Y, Z)U)X - g(X, Y)W^*(\xi, Z)U$$

$$+ \eta(Y)W^*(X, Z)U - g(X, Z)W^*(Y, \xi)U + \eta(Z)W^*(Y, X)U$$

$$- g(X, U)W^*(Y, Z)\xi + \eta(U)W^*(Y, Z)X]$$

$$+ b[S(X, W^*(Y, Z)U)\xi + \eta(W^*(Y, Z)U)\eta(X) - S(X, Y)W^*(\xi, Z)U$$

$$+ \eta(Y)W^*(QX, Z)U - S(X, Z)W^*(Y, \xi)U + \eta(Z)W^*(Y, QX)U$$

$$- S(X, U)W^*(Y, Z)\xi + \eta(U)W^*(Y, Z)QX] = 0. \tag{6.8}$$

Now, taking the inner product of above equation with $\xi$ and using equations (2.2), (2.4), (3.1), (3.4), (3.5) and (3.6), we obtain

$$A[-'R(Y, Z, U, X) - \frac{1}{2(n-1)}\{g(Z, U)S(X, Y) - g(Y, U)S(X, Z)$$

$$- 2S(Y, U)\eta(X)\eta(Z) - S(Z, U)g(X, Y) - S(Z, U)\eta(X)\eta(Y) - S(X, Z)\eta(Y)\eta(U)$$

$$+ S(X, Y)\eta(Z)\eta(U)\} + \frac{1}{2}\{g(Z, U)g(X, Y) - g(X, Z)g(Y, U)$$

$$+ g(X, Z)\eta(Y)\eta(U) - g(X, Y)\eta(Z)\eta(U)\}]$$

$$+ b[-'R(Y, Z, U, QX) + \frac{1}{2(n-1)}\{g(Z, U)S(X, QY) - (g(Y, U)S(X, QZ)$$

$$+ S(QX, Y)\eta(Z)\eta(U) - S(QX, Z)\eta(Y)\eta(U)\} + \frac{1}{2}\{g(Z, US(X, Y)$$

$$- g(Y, U)S(X, Z) + S(X, Z)\eta(Y)\eta(U) - S(X, Y)\eta(Z)\eta(U)\}]. \tag{6.9}$$

Taking a frame field and contraction over $Z$ and $U$, we get

$$S(QX, Y) = \left[(n-1)-\frac{A}{b}\right]S(X, Y)-\left[\frac{2(n-1)+r}{n}\right]\frac{A}{b}\eta(X)\eta(Y)$$

$$+\left[\frac{n(n-1)-r}{n}\right]\frac{A}{b}g(X, Y).$$

This completes the proof.

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# A NOTE ON IDEALS OF $C(X)$

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**ABSTRACT :** For a topological space $X$, let $\mathcal{P}$ be an ideal of closed sub-sets of $X$ and $C_{\mathcal{P}}(X)$ be the ideal of $C(X)$ of all functions $f$ such that the support of $f$ lies in $\mathcal{P}$. In this paper, we investigate the ideals of $C(X)$ which are of the form $C_{\mathcal{P}}(X)$ for some ideal $\mathcal{P}$ of closed sub-sets of $X$. We characterize $P$-spaces and almost $P$-spaces in terms of the ideals of the form $C_{\mathcal{P}}(X)$. Examples and counterexamples are given.

**Key words :** $C_{\mathcal{P}}(X)$, $C_K(X)$, $P$-space, almost $P$-space, $F$-space.

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## 1. INTRODUCTION

Throughout, $X$ will stand for a completely regular Hausdorff topological space, $C(X)$ denotes the ring of all real-valued continuous functions on $X$. For an $f \in C(X)$, $Z(f) = \{x \in X : f(x) = 0\}$ stands for the zero-set of $f$ and $cl_X(X - Z(f))$ stands for the support of $f$. Let $\mathcal{P}$ be a family of closed subsets of $X$ satisfying the following two conditions : ($i$) If $A, B \in \mathcal{P}$ then $A \cup B \in \mathcal{P}$. ($ii$) If $A \in \mathcal{P}$ and $B \subseteq A$ with $B$ closed in $X$ then $B \in \mathcal{P}$ i.e. $\mathcal{P}$ is an ideal of closed sets in $X$. In 2010, we initiated the ring $C_{\mathcal{P}}(X)$ for each ideal $\mathcal{P}$ of closed subsets of $X$ as $C_{\mathcal{P}}(X) = \{f \in C(X) : cl_X(X - Z(f)) \in \mathcal{P}\}$, [1]. It is clear that $C_{\mathcal{P}}(X)$ is a $z$-ideal (possibly improper) of $C(X)$, an ideal $I$ of $C(X)$ is called a $z$-ideal if $f \in I$, $Z(f) = Z(g)$ and $g \in C(X)$ imply that $g \in I$. It is also clear that if $\mathcal{P}$ denotes the family of all compact subsets of $X$ then $C_{\mathcal{P}}(X)$ coincides with $C_K(X)$ where $C_K(X) = \{f \in C(X) : cl_X(X - Z(f))$ is compact$\}$. Again if $\mathcal{P}$ denotes the family of all closed subsets of $X$ then $C_{\mathcal{P}}(X)$ coincides with $C(X)$.

**Lemma 1.1.** $C_{\mathcal{P}}(X) = C(X)$ if and only if $X \in \mathcal{P}$.

**Proof.** In fact, $C_{\mathcal{P}}(X) = C(X)$ if and only if $C_{\mathcal{P}}(X)$ contains units of $C(X)$ if and only if $X \in \mathcal{P}$.

**Notations 1.2.** (1) We denote the set of all ideals of closed sets in $X$ by $\Omega(X)$ and the family of all ideals of $C(X)$ which are of the form $C_{\mathcal{P}}(X)$ for some $\mathcal{P} \in \Omega(X)$ by $\mathcal{J}_{\Omega(X)}$.

It is clear that $\Omega(X)$ is closed with respect to arbitrary intersection.

(2) Suppose that $I$ is an ideal of $C(X)$. Consider the family of all members of $\Omega(X)$ containing $\{cl_X(X - Z(f)) : f \in I\}$. This family is nonempty since it contains the ideal of all closed subsets of $X$. Since $\Omega(X)$ is closed with respect to arbitary interesection, there exists a smallest member of $\Omega(X)$ containing $\{cl_X(X - Z(f)) : f \in I\}$ which we denote by $\mathcal{P}(I)$.

**Note 1.3.** It is obvious that $\{cl_X(X - Z(f)) : f \in I\}$ is closed with respect to finite union. Thus if $A \in \mathcal{P}(I)$ then $A \subseteq cl_X(X - Z(f))$ for some $f \in I$.

We now prove the following two lemmas.

**Lemma 1.4.** $\mathcal{J}_{\Omega(X)}$ is closed with respect to arbitrary intersection.

**Proof.** Let $\mathcal{J}_0 \subseteq \mathcal{J}_{\Omega(X)}$. Then $\mathcal{J}_0 = \{C_{\mathcal{P}}(X) : \mathcal{P} \in \Omega_0\}$ for some $\Omega_0 \subseteq \Omega(X)$. Since $\Omega(X)$ is closed with respect to arbitrary intersection, $\mathcal{P}_0 = \cap\{\mathcal{P} : \mathcal{P} \in \Omega_0\} \in \Omega(X)$. Also $\cap\mathcal{J}_0 = \cap\{C_{\mathcal{P}}(X) : \mathcal{P} \in \Omega_0\} = C_{\mathcal{P}0}(X)$. Since $\mathcal{P}_0 \in \Omega(X)$, we see that $\cap\mathcal{J}_0 \in \mathcal{J}_{\Omega(X)}$.

**Lemma 1.5.** Suppose $I$ is an ideal of $C(X)$. Then $C_{\mathcal{P}(I)}(X)$ is the smallest member of $\mathcal{J}_{\Omega(X)}$ containing $I$.

**Proof.** Obviously, $I \subseteq C_{\mathcal{P}(I)}(X)$. Let $I \subseteq C_{\mathcal{P}}(X)$ where $\mathcal{P} \in \Omega(X)$. Then $\{cl_X(X - Z(f)) : f \in I\} \subseteq \mathcal{P}$. Also $\mathcal{P}(I)$ is the smallest member of $\Omega(X)$ containing $\{cl_X(X - Z(f)) : f \in I\}$. Hence $\mathcal{P}(I) \subseteq \mathcal{P}$ and therefore $C_{\mathcal{P}(I)}(X) \subseteq C_{\mathcal{P}}(X)$.

**Corollary 1.6.** Suppose $I$ is an ideal of $C(X)$. Then $I \in \mathcal{J}_{\Omega(X)}$ if and only if $I = C_{\mathcal{P}(I)}(X)$.

As usual $\beta X$ denotes the Stone-Cech compactification of $X$. The maximal ideals of $C(X)$ are given by the family $\{M^p : p \in \beta X\}$ where $M^p = \{f \in C(X) : p \in cl_{\beta X}Z(f)\}$. Also for each $p \in \beta X$, the set $O^p = \{f \in C(X) : cl_{\beta X}Z(f)$ is a neighbourhood of $p\}$ is an ideal of $C(X)$. It is to be noted that $M^p$ and $O^p$ are $z$-ideals for all $p \in \beta X$. If $p \in X$ then we write $M_p$ and $O_p$ instead of $M^p$ and $O^p$ respectively. Thus $M_p = \{f \in C(X) : p \in Z(f)\}$ and $O_p = \{f \in C(X) : Z(f)$ is a neighbourhood of $p\}$. We now state the following theorem from Gillman-Jerison text, [7.12, [3]].

**Theorem 1.7.** Let $p \in \beta X$. Then $f \in O^p$ if and only if there is a neighbourhood $V$ of $p$ in $\beta X$ such that $V \cap X \subseteq Z(f)$.

We now prove the following theorem.

**Theorem 1.8.** For each $p \in \beta X$, $O^p \in \mathcal{J}_{\Omega(X)}$.

**Proof.** Let $p \in \beta X$ and $O^p = I$. Consider the ideal $C_{\mathcal{P}(I)}(X)$ and suppose $f \in C_{\mathcal{P}(I)}(X)$. Then $cl_X(X - Z(f)) \in \mathcal{P}(I)$ and so $cl_X(X - Z(f)) \subseteq cl_X(X - Z(g))$ for some $g \in I = O^p$ (Note 1.3). Hence $int_X Z(g) \subseteq int_X Z(f)$. Now since $g \in O^p$, by Theorem 1.7, we find an open set $V$ *in* $\beta X$ containing $p$ such that $V \cap X \subseteq Z(g)$. Since $V \cap X$ is open in $X$, we have $V \cap X \subseteq int_X Z(g)$. Thus $V \cap X \subseteq int_X Z(f)$. So by Theorem 1.7, $f \in O^p = I$. Hence $C_{\mathcal{P}(I)}(X) \subseteq I$ and therefore $I = C_{\mathcal{P}(I)}(X)$. Thus $O^p = I \in \mathcal{J}_{\Omega(x)}$.

A subset $Z$ of a space $X$ is called a zero-set if $Z = Z(f)$ for some $f \in C(X)$. A subset $A$ of a space $X$ is called regular closed if $A = cl_X int_X A$. Let us now prove the following theorems.

**Theorem 1.9.** Let $A \subseteq X$ be such that $cl_X A$ and $cl_X int_X A$ are both zero-sets in $X$. Then the following conditions are equivalent.

(1) $\cap_{p \in A} M_p = \cap_{p \in \text{int}_X cl_X A} O_p$.

(2) $\cap_{p \in A} M_p \in \mathcal{J}_{\Omega(x)}$.

(3) $cl_X A$ is regular closed.

**Proof.** (1) $\Rightarrow$ (2) Follows from Theorem 1.8 and Lemma 1.4.

(2) $\Rightarrow$ (3) : Put $cl_X A = B$ and choose two functions $f, g \in C(X)$ such that $Z(f) = B$ and $Z(g) = cl_X int_X B$. Then $int_X Z(f) = int_X B \subseteq int_X Z(g)$. Also $int_X Z(g) \subseteq int_X Z(f)$ since $Z(g) \subseteq Z(f)$. Therefore $int_X Z(f) = int_X Z(g)$ and hence $cl_X(X - Z(f)) = cl_X(X - Z(g))$. Now $f \in \cap_{p \in A} M_p$ since $A \subseteq Z(f)$. Also by assumption, $\cap_{p \in A} M_p = C_{\mathcal{P}}(X)$ for some $\mathcal{P} \in \Omega(X)$. Thus $f \in C_{\mathcal{P}}(X)$. So $cl_X(X - Z(f)) \in \mathcal{P}$ and hence $g \in C_{\mathcal{P}}(X)$ since $cl_X(X - Z(g)) = cl_X(X - Z(f))$. Now it is obvious that $\cap_{p \in A} M_p = \cap_{p \in cl_X A} M_p$. So $\cap_{p \in A} M_p = \cap_{p \in B} M_p$. Thus $g \in \cap_{p \in B} M_p$. Consequently, $B \cap Z(g)$ and therefore $B \subseteq cl_X int_X B$. Hence $B = cl_X int_X B$. Thus $B = cl_X A$ is regular closed.

(3) $\Rightarrow$ (1) : Obviously, $\cap_{p\in A}M_p \subseteq \cap_{p\in \text{int}_X cl_X A}O_p$. Suppose now that $f\in \cap_{p\in \text{int}_X cl_X A}O_p$. Hence $int_X cl_X A \subseteq Z(f)$ and thus $cl_X int_X cl_X A \subseteq Z(f)$. Since $cl_X A$ is regular closed, $cl_X A \subseteq Z(f)$. Thus $A \subseteq Z(f)$ and consequently, $f \in \cap_{p\in A}M_p$. Hence $\cap_{p\in A}M_p = \cap_{p\in \text{int}_X cl_X A}O_p$.

**Corollary 1.10.** Suppose $\{p\}$ is a zero-set in a space $X$. Then $M_p \in \mathcal{J}_{\Omega(X)}$ if and only if $p$ is an isolated point of $X$.

**Proof.** We note that $cl_X\{p\} = \{p\}$. Also $cl_X int_X cl_X\{p\} = \{p\}$ or $\emptyset$ according as $p$ is an isolated point or not. Thus if $\{p\}$ is a zero-set then $cl_X\{p\}$ and $cl_X int_X cl_X\{p\}$ both are zero-sets. Taking $A = \{p\}$, from Theorem 1.9 we now can say that $M_p \in \mathcal{J}_{\Omega(X)}$ if and only if $cl_X\{p\}$ is regulr closed i.e. if and only if $cl_X\{p\} = cl_X int_X cl_X\{p\}$ i.e. if and only if $p$ is an isolated point of $X$.

**Example 1.11.** The Corollary 1.10 becomes false if $\{p\}$ is not a zero-set in $X$. Take $X = [0, \omega_1]$, where $\omega_1$ is the first uncountable ordinal. Each $f \in C(X)$ is eventually constant on a tail $[\alpha, \omega_1]$ for some $\alpha < \omega_1$, hence $\{\omega_1\}$ is not a zero-set in $X$. But "$\{\omega_1\}$ is a $P$-point of $X$", [5O.1, [3]] and thus $M_{\omega_1} = O_{\omega_1}$. Consequently, by Theorem 1.8, $M_{\omega 1} \in \mathcal{J}_{\Omega(X)}$ although, $\omega_1$ is not an isolated point of $X$.

**Example 1.12.** Suppose $A = (0, 1)$, $B = [0, 1]$, $C = [0, 1] \cap \mathbb{Q}$ and $D = [0, 1] \cup \{2\}$. Then $cl_{\mathbb{R}}A$, $cl_{\mathbb{R}}B$, $cl_{\mathbb{R}}C$ are regular closed but $cl_{\mathbb{R}}D$ is not. So from Theorem 1.9 it follows that $\cap_{p\in A}M_p$, $\cap_{p\in B}M_p$, $\cap_{p\in C}M_p \in \mathcal{J}_{\Omega(\mathbb{R})}$ but $\cap_{p\in D}M_p \notin \mathcal{J}_{\Omega(\mathbb{R})}$. Again if $p \in \mathbb{R}$ then $M_p \notin \mathcal{J}_{\Omega(\mathbb{R})}$ as follows from Theorem 1.10.

## 2. *P*-SPACE, ALMOST *P*-SPACE, *F*-SPACE

A space $X$ is called a $P$-space if $M_p = O_p$ for each $p \in X$. Equivalently, $X$ is a $P$-space if every zero-set in $X$ is open. In 2010, we characterized $P$-spaces in the following theorem, [Theorem 5.4, [1]].

**Theorem 2.1.** A space $X$ is a $P$-space if and only if every ideal of $C(X)$ is of the form $C_{\mathcal{P}}(X)$ for some suitable family $\mathcal{P}$ of subsets of $X$ with $\mathcal{P} \in \Omega(X)$.

From Theorem 2.1 we can say that if $X$ is a $P$-space then each prime ideal of $C(X)$ is in $\mathcal{J}_{\Omega(X)}$. Interestingly, the converse is also true. In fact, if $X$ is not a $P$-space then $M_p \neq O_p$ for some $p \in X$. Hence there exists a prime ideal $P$ in $C(X)$ containing $O_p$ which is not a $z$-ideal, [4I-5, 6, [3]]. Thus $P \notin \mathcal{J}_{\Omega(X)}$ since each member of $\mathcal{J}_{\Omega(X)}$ is a $z$-deal. Hence we have the following theorem.

**Theorem 2.2.** For a space $X$, the following are equivalent.

(1) $X$ is a $P$-space.

(2) Every ideal of $C(X)$ is in $\mathcal{J}_{\Omega(X)}$.

(3) Every prime ideal of $C(X)$ is in $\mathcal{J}_{\Omega(X)}$.

A collection $\mathcal{F}$ of zero-sets in a space $X$ is called a $z$-filter on $X$ if (1) $\emptyset \notin \mathcal{F}$, (2) $\mathcal{F}$ is closed with respect to finite interesection and (3), $Z \in \mathcal{F}$ and $Z_1$ is a zero-set in $X$ with $Z_1 \supseteq Z$ imply that $Z_1 \in \mathcal{F}$, [2.2, [3]]. Recall that if $I$ is an ideal of $C(X)$ then the family $Z[I]$ $= \{Z(f) : f \in I\}$ is a $z$-filter on $X$, [2.3 (a), [3]]. A space $X$ is called an almost $P$-space if the interior of every nonempty zero-set in $X$ is nonempty. It is well-known that a space $X$ is an almost $P$-space if and only if every zero-set in $X$ is regular closed, [Proposition 1.1, [4]]. In the following theorem we characterize almost $P$-spaces.

**Theorem 2.3.** For a space $X$, the following are equivalent.

(1) $X$ is an almost $P$-space.

(2) $I \in \mathcal{J}_{\Omega(X)}$ for each $z$-ideal $I$ of $C(X)$.

(3) $M^p \in \mathcal{J}_{\Omega(X)}$ for each $p \in \beta X$.

(4) $M_p \in \mathcal{J}_{\Omega(X)}$ for each $p \in X$.

**Proof.** (1) $\Rightarrow$ (2) : Let $I$ be a $z$-ideal of $C(X)$. Suppose $f \in C_{\mathcal{P}(I)}(X)$. Then $cl_X(X - Z(f)) \in \mathcal{P}(I)$ and therefore $cl_X(X - Z(f)) \subseteq cl_X(X - Z(g))$ for some $g \in I$. Hence $int_X Z(g) \subseteq int_X Z(f)$ and so $cl_X int_X Z(g) \subseteq cl_X int_X Z(f)$. Since $X$ is an almost $P$-space, every zero-set in $X$ is regular closed and thus $Z(g) \subseteq Z(f)$. Also $g \in I$ shows that $Z(g) \in Z[I]$. Since $Z[I]$ is $z$-filter on $X$ we now have $Z(f) \in Z[I]$. Thus $Z(f) = Z(h)$ for some $h \in I$. Hence $f \in I$ since $I$ is a $z$-ideal. Thus $C_{\mathcal{P}(I)}(X) \subseteq I$ and so $I = C_{\mathcal{P}(I)}(X)$.

(2) $\Rightarrow$ (3) : Trivial since every maximal ideal in $C(X)$ is a $z$-ideal.

(3) $\Rightarrow$ (4) : Trivial.

(4) $\Rightarrow$ (1) : Suppose (1) is false. Then there is a nonempty zero-set, say $Z$ in $X$ such that $int_X Z = \emptyset$. Choose $p \in Z$ and suppose $Z = Z(f)$ where $f \in C(X)$. Then $f \in M_p$. Thus $cl_X(X - Z(f)) \in \mathcal{P}(I)$ where $I = M_p$. Now $cl_X(X - Z(f)) = X - int_X Z = X$ since $int_X Z = \emptyset$. Hence $X \in \mathcal{P}(I)$. Therefore $C_{\mathcal{P}(I)}(X) = C(X)$. thus $M_p = I \subsetneq C_{\mathcal{P}(I)}(X)$. From Corollary 1.6 it now follows that $M_p = I \notin \mathcal{J}_{\Omega(X)}$. Hence (4) is false.

We note that if $M^p \in \mathcal{J}_{\Omega(X)}$ for each $p \in \beta X - X$ then $X$ need not be an almost $P$-space. Consider the following example.

**Example 2.4.** Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Suppose $\Sigma = \mathbb{N} \cup \{\sigma\}$ where $\sigma \notin \mathbb{N}$. Define a topology on $\Sigma$ as follows : all points on $\mathbb{N}$ are isolated and the neighbourhoods of $\sigma$ are the sets $U \cup \{\sigma\}$ for $U \in \mathcal{U}$, [4M, [3]]. In $\Sigma$, the set $\{\sigma\}$ is a zero-set. Also $int_\Sigma\{\sigma\} = \emptyset$. So $\Sigma$ is not an almost $P$-space. Now choose $p \in \beta\Sigma - \Sigma$ and suppose $M^p = I$. If $M^p \subsetneq C_{\mathcal{P}(I)}(\Sigma)$ then $C_{\mathcal{P}(I)}(\Sigma) = C(\Sigma)$ since $M^p$ is maximal. Therefore $int_\Sigma Z(f) = \emptyset$ for some $f \in I = M^p$. Now $f \in M^p$ shows that $p \in cl_{\beta\Sigma}Z(f)$ and therefore $Z(f)$ is not compact. So $Z(f)$ contains points of $\mathbb{N}$. Since all point of $\mathbb{N}$ are isolated, it now follows that $int_\Sigma Z(f) \neq \emptyset$, a contradiction. Hence $M^p = C_{\mathcal{P}(I)}(\Sigma)$. Thus $M^p \in \mathcal{J}_{\Omega(\Sigma)}$ for each $p \in \beta\Sigma - \Sigma$.

An abstract ring $R$ is called an $F$-ring if each finitely generated ideal in $R$ is principal. A space $X$ is called an $F$-space if $C(X)$ is an F-ring. Equivalently, $X$ is an $F$-space if and only if for each $f \in C(X)$ there exists $k \in C(X)$ such that $f = k|f|$, [14.25, [3]]. In 2014, we characterized $F$-spaces in terms of the ideals $C_{\mathcal{P}}(X)$, [Theorem 2.1, [2]]. We now prove the following theorem.

**Theorem 2.5.** Consider the following conditions for a space $X$.

(1) Every finitely generated ideal in $C(X)$ is in $\mathcal{J}_{\Omega(X)}$.

(2) Every principal ideal in $C(X)$ is in $\mathcal{J}_{\Omega(X)}$.

(3) $X$ is an $F$-space.

Then (1) and (2) are equivalent and each of them implies (3).

**Proof.** (1) $\Rightarrow$ (2) : Trivial.

(2) $\Rightarrow$ (3) : Choose $f \in C(X)$. By (2), $(|f|) = C_{\mathcal{P}}(X)$ for some $\mathcal{P} \in \Omega(X)$. Hence $(|f|)$ is a $z$-ideal. Now $Z(|f|) = Z(f)$ shows that $f \in (|f|)$. Hence there exists $k \in C(X)$ such that $f = k|f|$. Hence $X$ is an $F$-space.

(2) $\Rightarrow$ (1) : If (2) is true then from (2) $\Rightarrow$ (3) above we see that $X$ is an $F$-space. Thus every finitely generated ideal in $C(X)$ is principal. Hence the proof follows.

It is to be noted that in Theorem 2.5, (3) need not imply (2) or (1). We consider the following example.

**Example 2.6.** Consider the space $\Sigma$ described in Example 2.4. It is an $F$-space, [4M-8, [3]]. Since $\{\sigma\}$ is a zero-set in $\Sigma$, we can select an $f \in C(\Sigma)$ such that $Z(f) = \{\sigma\}$. If possible now let $I = (f) \in \mathcal{I}_{\Omega(\Sigma)}$. Then $I = C_{\mathcal{P}}(\Sigma)$ for some $\mathcal{P} \in \Omega(\Sigma)$. Therefore $cl_{\Sigma}(\Sigma - Z(f)) \in \mathcal{P}$. Hence $cl_{\Sigma}(\Sigma - \{\sigma\}) \in \mathcal{P}$. Since $\sigma$ is not isolated in $\Sigma$, $cl_{\Sigma}(\Sigma - \{\sigma\}) = \Sigma$. Thus $\Sigma \in \mathcal{P}$ and consequently, $I = C_{\mathcal{P}}(\Sigma) = C(\Sigma)$, which is not possible since $f$ is not a unit in $C(\Sigma)$. Therefore $I = (f) \notin \mathcal{I}_{\Omega(\Sigma)}$.

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# A NOTE ON ROUGH STATISTICAL CONVERGENCE OF ORDER $\alpha$

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**ABSTRACT :** In this paper, in the line of Aytar[1] and Çolak [2], we introduce the notion of rough statistical convergence of order $\alpha$ in normed linear spaces and study some properties of the set of all rough statistical limit points of order $\alpha$.

**Key words and phrases :** Rough statistical convergence of order $\alpha$, rough statistical limit points of order $\alpha$,

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## 1. INTRODUCTION

The concept of statistical convergence was introduced by Steinhaus [9] and Fast [3] and later it was reintroduced by Schoenberg [8] independently. Over the years a lot of works have been done in this area. The concept of rough statistical convergence of single sequences was first introduced by S. Aytar [1]. Later the concept of statistical convergence of order $\alpha$ was introduced by R. Çolak [2].

If $x = \{x_n\}_{n \in N}$ is a sequence in some normed linear space $(X, \|\cdot\|)$ and $r$ is a nonnegative real number then $x$ is said to rough statistical convergent to $\xi \in X$ if for any $\varepsilon > 0$, $\lim_{n\to\infty} \frac{1}{n} \left|\left\{k \le n : \|x_k - \xi\| \ge r + \varepsilon\right\}\right| = 0$, [1].

For $r = 0$, rough statistical convergence coincides with statistical convergence.

In this paper following the line of Aytar [1] and Çolak [2] we introduce the notion of rough statistical convergence of order $\alpha$ in normed linear spaces and prove some properties of the set of all rough statistical limit points of order $\alpha$.

## 2. BASIC DEFINITIONS AND NOTATONS

**Definition 2.1.** Let $K$ be a subset of the set of positive integers N. Let $K_n = \{k \in K : k \leq n\}$. Then the natural density of $K$ is given by $\lim_{n\to\infty} \frac{|K_n|}{n}$, where, $|K_n|$ denotes the number of elements in $K_n$.

**Definition 2.2.** Let $K$ be a subset of the set of positive integers N and $\alpha$ be any real number with $0 < \alpha \leq 1$. Let $K_n = \{k \in K : k \leq n\}$. Then the natural density of order $\alpha$ of $K$ is given by $\lim_{n\to\infty} \frac{|K_n|}{n^\alpha}$, where $|K_n|$ denotes the number of elements in $K_n$.

**Note 2.1.** *Let $x = \{x_n\}_{n\in N}$ be a sequence. Then $X$ satisfies some property $P$ for all $k$ except a set whose natural density is zero. Then we say that the sequence $x$ satisfies $P$ for almost all $k$ and we abbreviated this by a.a.k.*

**Note 2.2.** *Let $x = \{x_n\}_{n\in N}$ be a sequence. Then $x$ satisfies some property $P$ for all $k$ except a set whose natural density of order $\alpha$ is zero. Then we say that the sequence $x$ satisfies $P$ for almost all $k$ and we abbreviated this by a.a.$k(\alpha)$.*

**Definition 2.3.** Let $x = \{x_n\}_{n\in N}$ be a sequence in a normed linear space $(X, \|\cdot\|)$ and $r$ be a nonncgative real number. Let $0 < \alpha \leq 1$ be given. Then $x$ is said to be rough statistical convergent of order $\alpha$ to $\xi \in X$, denoted by $x_n \xrightarrow{st-r^\alpha} \xi$ *if for any* $\varepsilon > 0$, $\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n : \|x_k - \xi\| \geq r + \varepsilon\}| = 0$, that is a.a.k.$(\alpha)$ $\|x_k - \xi\| < r + \varepsilon$ for every $\varepsilon > 0$ and some $r > 0$. In this case $\xi$ is called a $r^\alpha$-st-limit of $x$.

The set of all rough statistical convergent sequences of order $\alpha$ will be denoted by $rS^\alpha$ for fixed $r$ with $0 < r \leq 1$.

Throughout this paper $X$ will denote a normed linear space and $r$ will denote a nonnegative real number and $x$ will denote the sequence $x = \{x_n\}_{n\in N}$ in $X$.

In general, the $r^\alpha$-st-limit point of a sequence may not be unique. So we consider $r^\alpha$-st-limit set of a sequence $x$, which is defined by $st\text{-}LIM_x^{r^\alpha} = \{\xi \in X : x_n \xrightarrow{st-r^\alpha} \xi\}$. The

sequence $x$ is said to be $r^\alpha$-statistical convergent provided that $st\text{-}LIM_x^{r^\alpha} = \emptyset$. For unbounded sequence rough limit set $LIM_x^r = \emptyset$.

But in case of rough statistical convergence of order α $st\text{-}LIM_x^{r^\alpha} = \emptyset$ even though the sequence is unbounded. For this we consider the following example.

**Example 2.1.** *Let $X = \mathbb{R}$. We define a sequence in the following way,*

$$x_n = (-1)^n : i \neq n^2, \alpha = 1$$
$$= n, \text{ otherwise}$$

*Then*

$$st\text{-}LIM_x^{r^\alpha} = \emptyset, \text{ if } r < 1$$
$$= [1 - r, r - 1], \text{ otherwise.}$$

*and* $LIM_x^{r^\alpha} = \emptyset$ for all $r \geq 0$.

**Definition 2.4.** A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be statistically bounded if there exists a positive real number $M$ such that $\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : \| x_k \| \geq M\}| = 0$

**Definition 2.5.** Let $0 < \alpha \leq 1$ be given. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be statistically bounded of order α if there exists a positive real number $M$ such that $\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : \| x_k \| \geq M\}| = 0$.

## 3. MAIN RESULTS

**Theorem 3.1.** *Let $x$ be a sequence in $X$. Then $x$ is statistically bounded of order if and only if there exists a nonegative real number $r$ such that $st\text{-}LIM_x^{r^\alpha} = \emptyset$.*

**Proof.** The condition is necessary.

Since the sequence $x$ is statistically bounded, there exists a positive real number $M$ such

that $\lim_{n\to\infty}\frac{1}{n^\alpha}|\{k \le n : \| x_k \| \ge M\}| = 0$. Let $K = \{k \in \mathbb{N} : \| x_k \| \ge M\}$. Define $r' = sup\{\| x_k \| : k \in K^c\}$. Then the set $st\text{-}LIM_x^{r'^\alpha}$ contains the origin of $X$. Hence $st\text{-}LIM_x^{r^\alpha} = \emptyset$.

The condition is sufficient.

Let $st\text{-}LIM_x^{r^\alpha} = \emptyset$ for some $r \ge 0$. Then there exists $l \in X$ such that $l \in st\text{-}LIM_x^{r^\alpha}$. *Then* $\lim_{n\to\infty}\frac{1}{n^\alpha}|\{k \le n : \| x_k - l \| \ge r + \varepsilon\}| = 0$ for each $\varepsilon > 0$. Then we say that almost all $x_k$'s are contained in some ball with any radius greater than $r$. So the sequence $x$ is statistically bounded.

**Theorem 3.2.** *If* $x' = \{x_{n_k}\}_{n\in\mathbb{N}}$ *is a subsequence of* $x = \{x_n\}_{n\in\mathbb{N}}$ *then* $st\text{-}LIM_x^{r^\alpha} \subseteq st\text{-}LIM_{x'}^{r^\alpha}$.

**Proof.** The proof is straight forward. So we omit it.

**Theorem 3.3.** $st\text{-}LIM_x^{r^\alpha}$*, the rough statistical limit set of order* $\alpha$ *of a sequence* $x$ *is closed.*

**Proof.** If $st\text{-}LIM_x^{r^\alpha} = \emptyset$ then there is nothing to prove. So we assume that $st\text{-}LIM_x^{r^\alpha} = \emptyset$. We can choose a sequence $\{y_k\}_{n\in\mathbb{N}} \subsetneq st\text{-}LIM_x^{r^\alpha}$ such that $y_k \to y_*$ for $k \to \infty$. It sufficcs to prove that $y_* \in st\text{-}LIM_x^{r^\alpha}$.

Let $\varepsilon > 0$. Since $y_k \to y_*$ there exists $k_\varepsilon \in \mathbb{N}$ such that $\| y_k - y_* \| < \frac{\varepsilon}{2}$ for $k > k_\varepsilon$. Now choose $k_0 \in \mathbb{N}$ such that $k_0 > k_\varepsilon$. Then we can write $\| y_{k0} - y_* \| < \frac{\varepsilon}{2}$. Again since $\{y_k\}_{n\in\mathbb{N}} \subsetneq st\text{-}LIM_x^{r^\alpha}$ we have $y_{k0} \in st\text{-}LIM_x^{r^\alpha}$. This implies

$$\lim_{n\to\infty n^\alpha}\frac{1}{n^\alpha}\left|\left\{k \le n : \left\| x_k - yk_0 \right\| \ge r + \tfrac{\varepsilon}{2}\right\}\right| = 0 \tag{1}$$

Now we show the inclusion

$$\left\{k \le n : \left\| x_k - yk_0 \right\| < r + \tfrac{\varepsilon}{2}\right\} \subseteq \left\{k \le n : \left\| x_k - y_* \right\| < r + \varepsilon\right\} \tag{2}$$

holds.

Choose $j \in \left\{k \leq n : \left\| x_k - y_{k_0} \right\| < r + \frac{\varepsilon}{2}\right\}$. Then we have $\left\| x_j - y_{k_0} \right\| < r + \frac{\varepsilon}{2}$ and hence $\left\| x_j - y_* \right\| \leq \left\| x_j - y_{k_0} \right\| + \left\| y_{k_0} - y_* \right\| < r + \varepsilon$ which implies $j \in \{k \leq n : \|x_k - y_*\| < r + \varepsilon\}$, which proves the inclusion (2).

From (1) we can say that the set on the right hand side of (2) has natural density 1. Then the set on the left hand side of (2) must have natural density 1. Hence we get $\lim\limits_{n\to\infty} \frac{1}{n^\alpha} \left| \left\{k \leq n : \left\| x_k - y_* \right\| \geq r + \varepsilon\right\} \right| = 0$.

This completes the proof. □

**Theorem 3.4.** *Let $x$ be a sequence in $X$. Then the rough statistical limit set of order $\alpha$ $st\text{-}LIM_x^{r^\alpha}$ is convex.*

**Proof :** Choose $y_1, y_2 \in st\text{-}LIM_x^{r^\alpha}$ and let $\varepsilon > 0$. Define $K_1 = \{k \leq n : \|x_k - y_1\| \geq r + \varepsilon\}$ and $K_2 = \{k \leq n : \| x_k - y_2 \| \geq r + \varepsilon\}$. Since $y_1, y_2 \in st\text{-}LIM_x^{r^\alpha}$, we have $\lim\limits_{n\to\infty} \frac{1}{n^\alpha}|K_1| = \lim\limits_{n\to\infty} \frac{1}{n^\alpha}|K_2| = 0$. Let $\lambda$ be any positive real number with $0 \leq \lambda \leq 1$.

Then

$$\| x_k - [(1 - \lambda)y_1 + \lambda y_2] \| = \| (1 - \lambda)(x_k - y_1) + \lambda(x_k - y_2) \| < r + \varepsilon$$

for each $k \in K_1^c \cap K_2^c$. Since $\lim\limits_{n\to\infty} \frac{1}{n^\alpha} \left| K_1^c \cap K_2^c \right| = 1$, we get

$$\lim_{n\to\infty} \frac{1}{n^\alpha} \left| \left\{ k \leq n : \left\| x_k - \left[(1-\lambda)y_1 + \lambda y_2\right] \right\| \geq r + \varepsilon \right\} \right| = 0$$

that is

$$[(1 - \lambda)y_1 + (\lambda)y_2] \in st\text{-}LIM_x^{r^\alpha}$$

which proves the convexity of the set $st\text{-}LIM_x^{r^\alpha}$. □

**Theorem 3.5.** *Let $x$ be a sequence in $X$ and $r > 0$. Then the sequence $x$ is rough statistical convergent of order $\alpha$ to $\xi \in X$ if and only if there exists a sequence $y = \{y_n\}_{n\in\mathbb{N}}$ in $X$ such that $y$ is statistically convergent of order $\alpha$ to $\xi$ and $\|x_n - y_n\| \le r$ for all $n \in \mathbb{N}$.*

**Proof.** The condition is necessary.

Let $x_n \xrightarrow{st-r^\alpha} \xi$. Choose any $\varepsilon > 0$. Then we have

$$\lim_{n\to\infty}\frac{1}{n^\alpha}\left|\left\{k \le n : \|x_k - \xi\| \ge r + \varepsilon\right\}\right| = 0 \text{ for some } r > 0. \tag{3}$$

Now we define

$$\begin{aligned} y_n &= \xi, \text{ if } \|x_n - \xi\| \le r \\ &= x_n + r\frac{\xi - x_n}{\|x_n - \xi\|}, \text{ otherwise} \end{aligned}$$

Then we can write

$$\begin{aligned} \|y_n - \xi\| &= 0, \text{ if } \|x_n - \xi\| \le r \\ &= \|x_n - \xi\| - r, \text{ otherwise} \end{aligned}$$

and by definition of $y_n$, we have $\|x_n - y_n\| \le r$ for all $n \in \mathbb{N}$. Hence by (3) and the definition of $y_n$ we get $\lim_{n\to\infty}\frac{1}{n^\alpha}\left|\left\{k \le n : \|y_k - \xi\| \ge \varepsilon\right\}\right| = 0$. Which implies that the sequence $\{y_n\}_{n\in\mathbb{N}}$ is statistically covergent of order $\alpha$ to $\xi$.

The condition is sufficient.

Since $\{y_n\}_{n\in\mathbb{N}}$ is statistically convergent of order $\alpha$ to $\xi$, we have $\lim_{n\to\infty}\frac{1}{n^\alpha}\left|\left\{k \le n : \|y_k - \xi\| \ge \varepsilon\right\}\right| = 0$ for all $\varepsilon > 0$. Also since for a given $r > 0$ and for the sequence $x = \{x_n\}_{n\in\mathbb{N}}$ $\|x_n - y_n\| < r$, the inclusion $\{k \le n : \|x_k - \xi\| \ge r + \varepsilon\} \subseteq \{k \le n : \|y_k - \xi\| \ge \varepsilon\}$ holds. Hence we get $\lim_{n\to\infty}\frac{1}{n^\alpha}\left|\left\{k \le n : \|x_k - \xi\| \ge r + \varepsilon\right\}\right| = 0$. This completes the proof. □

**Theorem 3.6.** *For an arbitrary $c \in \Gamma_x$, where $\Gamma_x$ is the set of all rough statistical cluster points of a sequence $x \in X$, we have for a positive real number $r$, $\| \xi - c \| \leq r$ for all $\xi \in st\text{-}LIM_x^{r^\alpha}$.*

**Proof.** Let $0 < \alpha \leq 1$ be given. On the contrary let assume that there exists a point $c \in \Gamma_x$ and $\xi \in st\text{-}LIM_x^{r^\alpha}$ such that $\| \xi - c \| > r$. Choose $\varepsilon = \frac{\|\xi - c\| - r}{3}$. Then

$$\{k \leq n : \| x_k - \xi \| \geq r + \varepsilon\} \supseteq \{k \leq n : \| x_k - c \| < \varepsilon\} \tag{4}$$

holds. Since $c \in \Gamma_x$, we have $\lim_{n\to\infty} \frac{1}{n^\alpha} \left|\{k \leq n : \|x_k - c\| < \varepsilon\}\right| \neq 0$. Hence by (4) we have $\lim_{n\to\infty} \frac{1}{n^\alpha} \left|\{k \leq n : \|x_k - \xi\| < r + \varepsilon\}\right| \neq 0$. This is a contradictin to the fact $\xi \in st\text{-}LIM_x^{r^\alpha}$.

**Theorem 3.7.** *Let $x$ be sequence in the strictly convex space. Let $r$ and $\alpha$ be two positive real numbers. If for any $y_1, y_2 \in st\text{-}LIM_x^{r^\alpha}$ with $\| y_1 - y_2 \| = 2r$, then $x$ is statistically convergent of order $\alpha$ to $\frac{y_1 + y_2}{2}$.*

**Proof.** Let $z \in \Gamma_x$. Then for any $y_1, y_2$ $st\text{-}LIM_x^{r^\alpha}$ implies

$$\| y_1 - z \| \leq r \text{ and } \| y_2 - z \| \leq r. \tag{5}$$

On the other hand we have

$$2r = \| y_1 - y_2 \| \leq \| y_1 - z \| + \| y_2 - z \|. \tag{6}$$

Hence by (5) and (6) we get $\| y_1 - z \| = \| y_2 - z \| = r$. Since

$$\tfrac{1}{2}(y_2 - y_1) = \tfrac{1}{2}[(z - y_1) + (y_2 - z)] \tag{7}$$

and $\| y_1 - y_2 \| = 2r$, we get $\|\frac{1}{2}(y_2 - y_1)\| = r$. By strict convexity of the space and from the equality (7) we get $\frac{1}{2}(y_1 - y_2) = z - y_1 = y_2 - z$, which implics that $z = \frac{1}{2}(y_1 + y_2)$. Hence $z$ is unique statistical cluster point of the sequence $x$. On the other hand, from the assumption

$y_1, y_2 \in st\text{-}LIM_x^{r^\alpha}$ implies that $st\text{-}LIM_x^{r^\alpha} \neq \emptyset$. So by Theorem 3.1 the sequence $x$ is statistically bounded of order $\alpha$. Since $z$ is the unique statistical cluster point of the statistically bounded sequence $x$ of order $\alpha$ we have the sequence $x$ is statistically convergent to $z = \frac{1}{2}(y_1 + y_2)$. □

**Theorem 3.8.** *Let* $0 < \alpha \leq 1$ *and* $x$ *and* $y$ *be two sequences. Then*

*(i) if* $r^\alpha$-$st$-$\lim x = x_0$ *and* $c \in \mathbb{R}$, *then* $r^\alpha$-$st$-$\lim cx = cx_0$

*(ii)* if $r^\alpha$-$st$-$\lim x = x_0$ *and* $r^\alpha$-$st$-$\lim y = y_0$ *then* $r^\alpha$-$st$-$\lim(x + y) = x_0 + y_0$.

**Proof.** (i) If $c = 0$ it is trivial. Suppose that $c \neq 0$. Then the proof of (i) follows from $\frac{1}{n^\alpha}|\{k \leq n : \| cx_k - cx_0 \| \geq r + \varepsilon\}| \leq \frac{1}{n^\alpha}|\{k \leq n : \|x_k - x_0\| \geq \frac{r+\varepsilon}{|c|}\}| = \frac{1}{n^\alpha}|\{k \leq n : \|x_k - x_0\| \geq \frac{r}{|c|} + \frac{\varepsilon}{|c|}\}|$. Since $x$ is rough statistical convergent of order $\alpha$, hence $cx$ is also rough statistical convergent of order $\alpha$.

Again

$$\frac{1}{n^\alpha}|\{k \leq n : \|(x_k + y_k) - (x_0 + y_0)\| \geq r + \varepsilon\}| \leq \frac{1}{n^\alpha}|\{k \leq n : \| x_k - x_0 \| \geq r + \varepsilon\}| + \frac{1}{n^\alpha}|\{k \leq n : \| y_k - y_0 \| \geq r + \varepsilon\}|$$

It is casy to see that every convergent sequence is rough statistical convergent of order $\alpha$, but the converse is not true always.

**Example 3.1.** *Let us consider the following sequence of real numbers defined by,*

$$x_k = 1, \text{ if } k = n^3$$
$$= 0, \text{ Otherwise}$$

*Then it is easy to see that the sequence is rough statistical convergent of order* $\alpha$ *with* $rS^\alpha$-$\lim x_k = 0$ *for* $\alpha > \frac{1}{3}$, *but it is not a convergent sequence.*

**Theorem 3.9.** *Let* $0 < \alpha \leq \beta \leq 1$. *Then* $rS^{\alpha} \subseteq rS^{\beta}$, *where* $rS^{\alpha}$ *and* $rS^{\beta}$ *denote the set of all rough statistical convergent sequence of order* $\alpha$ *and* $\beta$ *respectively.*

**Proof.** If $0 < \alpha \leq \beta \leq 1$ then

$$\frac{1}{n^{\beta}}|\{k \leq n : \| x_k - l \| \geq r + \varepsilon\}| \leq \frac{1}{n^{\alpha}}|\{k \leq n : \| x_k - l \| \geq r + \varepsilon\}|$$

for every $\varepsilon > 0$ and some $r > 0$ with limit $l$.

Clearly this shows that $rS^{\alpha} \subseteq rS^{\beta}$.

We do not know whether for a sequence $x$ in $X$, $r > 0$ and for $0 < \alpha < 1$, $diam\left(st\text{-}LIM_x^{r^{\alpha}}\right)$ $\leq 2r$ is true or not.

So we leave this above fact as an open problem.

**Open Problem 3.1.** Is it true for a sequence $x$ in $X$, $r > 0$ and $0 < \alpha < 1$, $diam\left(st\text{-}LIM_x^{r^{\alpha}}\right)$ $\leq 2r$.

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# PROPERTIES OF δ-LORENTZIAN β-KENMOTSU MANIFOLDS

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**ABSTRACT :** In this paper, we obtain basic results and properties of δ-Lorentzian β-Kenmotsu manifolds. Properties of Ricci semisymmetric and conformally flat δ-Lorentzian β-Kemmotsu manifolds are obtained.

**Key words and phrases :** Sasakian manifold, Lorentzian α-Sasakian manifold, Ricci semisymmetric manifold, conformal curvature tensor. Date : August 5, 2015.

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## 1. INTRODUCTION

In 1969, T. Takahashi [12] has introduced Sasakian manifolds with pseudo Riemannian metric and showed that one can study the Lorentzian Sasakian structure with an indefinite metric. In 1990, K.L. Duggal [8] has initiated the space time manifolds. S.Y. Perktas, Erol Kilic, M.M. Tripathi and S. Keles [10] and Hakan Oztur, Nasip Aktan and Cengizhan Murathan [9], U.C. De [3], U.C. De and J.B. Jun, Goutam Pathak [4], U.C. De [5], Lovejoy Das, R.N. Singh, Manoj Kumar Pande [6], U.C. de, A. Yildiz, B.E. Acet [7] have studied the various properties of Lorentzian β-Kenmotsu manifolds. Some other authors (see the list [2], [8], [11]) studied Lorentzian β-Kenmotsu manifolds. Recently Vilas Khairnar [15] studied weak symmetries of δ-Lorentzian β-Kenmotsu manifolds.

In Section 2, we consider $(2n+1)$ dimensional differrentiable manifold M with Lorentzian almost contact metric structure with indefinite metric g. In this Section, some background information for defining δ-Lorentzian β-Kenmotsu manifold has been given. Further, various basic results are studied. Concrete Example for the existance of δ-Lorentzian β-Kenmotsu manifold is given. Section 3 is devoted to the study of the generalised recurrent properties of δ-Lorentzian β-Kenmotsu manifolds. This section includes some of the results of Hakan Oztur, Nasip Aktan and Cengizhan Murathan [9] as special cases.

Section 4 deals with the properties of Ricci semisymmetric and semisymmetric $\delta$-Lorentzian $\beta$-Kenmotsu manifolds and generalises the results of Hakan Oztur, Nasip Aktan and Cengizhan Murathan [9]. Finally, in Section 5, the properties of confomally flat space $\delta$-Lorentzian $\beta$-Menmotsu manifold are obtained.

## 2. $\delta$ - LORENTZIAN $\beta$ - KENMOTSU MANIFOLD

For an almost Lorentzian contact manifold, we have

$$\varphi^2 X = X + \eta(X)\xi,\ \eta(\xi) = -1,\ \eta(X) = g(X, \xi)$$

where $\varphi$ is a tensor field of type (1,1), $\xi$ is characteristic vector field and $\eta$ is the 1-form. From these conditions, one can deduce that

$$\varphi(\xi) = 0,\ \eta(\varphi(X)) = 0$$

for any vector field $X$ on $M$. It is well known that the Lorentzian contact metric structure or Lorentzian Kenmotsu structure [8] satisfies

$$(\nabla_X \varphi)Y = g(\varphi(X), Y)\xi + \eta(Y)\varphi(X)$$

for any $C^\infty$ vector fields $X$ and $Y$ on $M$.

$$(\nabla_X \varphi)Y = \beta\{g(\varphi(X), Y)\xi + \eta(Y)\varphi(X)\}$$

for any $C^\infty$ vector fields $X$ and $Y$ on $M$ and $\beta$ is a nonzero constant on $M$. Using above formula, one can deduce for a $\beta$-Kenmotsu manifolds

$$\nabla_X \xi = \beta\{X + \eta(X)\xi\}$$

and

$$(\nabla_X \eta)(Y) = \beta\{g(X, Y) + \eta(X)\eta(Y)\}$$

**Definition 2.1.** A differentiable manifold M of dimension $(2n + 1)$ is called a $\delta$-Lorentzian manifold if it admits a (1, 1) tensor field $\varphi$, a contravariant vector field $\xi$, a covariant vector field $\eta$ and an indefinite metric $g$ which satisfy

$$\varphi^2 X = X + \eta(X)\xi,\ \eta(\xi) = -1,\ \eta(\varphi(X)) = 0 \tag{2.1}$$

$$g(\xi, \xi) = -\delta,\ \eta(X) = \delta g(X, \xi) \tag{2.2}$$

(2.3) $$g(\varphi(X), \varphi(Y)) = g(X, Y) + \delta\eta(X)\eta(Y),$$

where $\delta$ is such that $\delta^2 = 1$ and for any vector field $X$, $Y$ on $M$. The structure defined above is called a δ-Lorentzian almost contact metric structureso. Manifold $M$ togather with the structure $(\varphi, \xi, \eta, g, \delta)$ is called a δ-Lorentzian Kenmotsu manifold if

$$(\nabla_X\varphi)Y = g(\varphi(X), Y)\xi + \delta\eta(Y)\varphi(X)$$

**Definition 2.2.** A δ-Lorentzian almost contact metric manifold $M(\varphi, \xi, \eta, g, \delta)$ is called a *δ-Lorentzian β-Kenmotsu manifold* if

(2.4) $$(\nabla_X\varphi)(Y) = \beta\{g((\varphi(X), Y)\xi + \delta\eta(Y)\varphi(X)\},$$

where $\nabla$ is the Levi-Civita connection with respect to g, $\beta$ is a smooth function on $M$ and $X$, $Y$ are any vector fields on $M$ and $\delta$ is such that $\delta^2 = 1$.

If $\delta = 1$, then δ-Lorentzian β-Kenmotsu manifold is the usual ***Lorentzian β-Kenmotsu manifold*** and is called the time like manifold. In this case, $\xi$ is called a time like vector field.

**Example 2.1.** Let $M$ be a δ-Lorentzian β-Kenmotsu manifold. We put

$$\overline{\varphi} = \varphi, \ \overline{\xi} = -\xi, \ \overline{\eta} = -\eta, \ \overline{g} = -g, \ \overline{\delta} = -\delta$$

Then $\left(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}, \overline{\delta}\right)$ defines a δ-Lorentzian almost contact structure on $M$. For,

$$\overline{\varphi}^2 X = X + \overline{\eta}(X)\xi, \ \overline{g}(\overline{\xi}, \overline{\xi}) = -\overline{\delta}, \ \overline{\eta}(\overline{\xi}) = -1, \ \overline{\eta}(X) = \overline{\delta}\overline{g}(X, \overline{\xi})$$

$$\overline{g}(\overline{\varphi}(X), \overline{\varphi}(Y) = \overline{g}(X, Y) + \overline{\delta}\overline{\eta}(X)\overline{\eta}(Y)$$

which by Definition 2.2, $\left(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}, \overline{\delta}\right)$ is a δ Lorentzian almost contact metric structure and further, it is a δ Lorentzian contact metric structure on $M$. Thus we conclude that if $(\varphi, \xi, \eta, g, \delta)$ is a δ-Lorentzian contact metric structure on $M$, then $\left(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}, \overline{\delta}\right)$ is also a δ Lorentzian contact metric structure on $M$.

Suppose $(\varphi, \xi, \eta, g, \delta)$ is a δ-Lorentzian normal contact metric structure on $M$. Since the parallelism with respect to $g$ and $\overline{g}$ are the same, we get

$$(\overline{\nabla}_X\overline{\varphi})(Y) = (\nabla_X\varphi)(Y) = g(\varphi(X), Y)\xi + \delta\eta(Y)\varphi(X)$$

from which, finally we have

$$(\overline{\nabla}_X\overline{\varphi})(Y) = \overline{g}(\overline{\varphi}(X), Y)\overline{\xi} + \overline{\delta}\overline{\eta}(Y)\overline{\varphi}(X)$$

which shows that $\left(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}, \overline{\delta}\right)$ is also a $\delta$- Lorentzian normal contact metric structure on $M$. Similar arguments hold for $\delta$-Lorentzian $\beta$-Kenmotsu manifolds.

**Lemma 2.1.** *For a $\delta$-Lorentzian $\beta$-Kenmotsu manifolds, we have*

(2.5) $$\nabla_X\xi = \delta\beta\{X + \eta(X)\xi\}$$

*for any vector field $X$ on $M$.*

**Proof.** From (2.4) of Definition 2.2, we have

$$\nabla_X(\varphi(Y) - \varphi(\nabla_X Y) = \beta\{g(\varphi(X), Y)\xi + \delta\eta(Y)\varphi(X)\}$$

Now taking $Y = \xi$ in the above equation and using (2.1), we get

$$-\varphi(\nabla_X Y) = -\beta\delta\varphi(X)$$

Applying $\varphi$ on both sides of the above equation and using the fact that $(\nabla_X g)(\xi, \xi) = 0$ and (2.1), we get (2.5)

**Example 2.2.** Let us consider the 3-dimensional manifold $M^3 = \{(x, y, z) \in R^3\}$, where $(x, y, z)$ are the standered co-ordinates in $R^3$. Following vector fields are linearly independent at each point of $M^2$.

$$e_1 = f(z)\frac{\partial}{\partial x} + g(z)\frac{\partial}{\partial y}, \quad e_2 = -g(z)\frac{\partial}{\partial x} + f(z)\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}$$

where $f$ and $g$ are given by

$$f = ae^{-\beta z}$$

$$g = be^{-\beta z}$$

with $f^2 + g^2 \neq 0$ for constants $a$ and $b$ and $\beta$. Let $g$ be an indefinite metric defined by

$$g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0,\ g(e_e, e_1) = g(e_2, e_2) = 1,\ g(e_3, e_3) = -\delta$$

and the δ-Lorentzian metrc $g$ is thus given by

$$g = \frac{1}{f^2 + g^2}\left\{(dx)^2 + (dy)^2 - \delta(dz)^2\right\}$$

where $\delta = \pm 1$. If $\delta = -1$, then δ-Lorentzian metric $g$ becomes a Riemannian positive definite metric on $M$ so that in this case, the characteristic vector field $\xi$ becomes a space like and if $\delta = 1$, then it becomes a light like.

Let $\eta$ be the 1-form defined by

$$\eta(X) = \delta g(X, \xi)$$

for any vector field on $M^3$. Let $\varphi$ be the tensor field of type (1,1) defined by

$$\varphi(e_1) = -e_1,\ \varphi(e_2) = -e_2,\ \varphi(e_3) = 0$$

Using the linearity property of $g$ and $\varphi$, one can deduce

$$\varphi^2 X = X + \eta(X)\xi,\ \eta(X) = -1,\ g(\xi, \xi) = -\delta,\ g(\varphi(X), \varphi(Y)) = g(X, Y) + \delta\eta(X)\eta(Y)$$

Also

$$\eta(e_1) = 0,\ \eta(e_2) = 0,\ \eta(e_3) = -1$$

for any vector field $X$ and $Y$ on $M$. Let $\nabla$ be the Levi-Civita connection with respect to $g$. Then we have

$$[e_1, e_2] = 0,\ [e_1, e_3] = \delta\beta e_1,\ [e_2, e_3] = \delta\beta e_2$$

Using Koszule's formulas for Levi-Civita connection $\nabla$ with respect to $g$, that is

$$2g(\nabla_X, Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y)$$
$$- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y])$$

one can easily calculate

$$\nabla_{e_1} e_3 = \delta\beta_{e_1},\ \ \nabla_{e_3} e_3 = 0,\ \ \nabla_{e_2} e_3 = \delta\beta e_2$$

$$\nabla_{e_2} e_2 = \delta\beta_{e_3},\ \ \nabla_{e_1} e_2 = 0,\ \ \nabla_{e_2} e_1 = 0$$

$$\nabla_{e_1} e_1 = \delta\beta_{e_3},\ \ \nabla_{e_3} e_2 = \delta\beta e_2,\ \ \nabla_{e_3} e_1 = \delta\beta e_1$$

With these information, the structure $(\eta, \xi, \eta, g, \delta)$ satisfies (2.4) and (2.5). Hence $M^3(\varphi, \xi, \eta, g, \delta)$ defines a $\delta$-Lorentzian $\beta$-Kenmotsu manifold.

**Lemma 2.2.** *for a $\delta$-Lorentzian $\beta$-Kenmotsu manifold $M$. we have*

$$(\nabla_X \eta)(Y) = \beta\{g(X, Y) + \delta\eta(X)\eta(Y)\} \tag{2.6}$$

*for any vector fields $X$ and $Y$ on $M$.*

**Proof.** Consider,

$$\begin{aligned}(\nabla_X \eta)(Y) &= \nabla_X(\eta(Y)) - \eta(\nabla_X Y) \\ &= \delta\nabla_X(g(Y, \xi)) - \delta g(\nabla_X Y, \xi) \\ &= \delta g(Y, \nabla_X \xi)\end{aligned} \tag{2.7}$$

Using (2.5) in (2.7) we get (2.6).

**Lemma 2.3.** *For a $\delta$-Lorentzian $\beta$-Kenmotsu manifold $M$, we have*

$$R(X, Y)\xi = \beta^2\{\eta(Y)X - \eta(X)Y\} + \delta\{(X\beta)\varphi^2 Y - (Y\beta)\varphi^2 X\} \tag{2.8}$$

$$R(\xi, Y)\xi = \{\beta^2 + \delta(\xi\beta)\}\varphi^2 Y,\ R(\xi, \xi)\xi = 0 \tag{2.9}$$

*for any vector fields $X$ and $Y$ on $M$.*

**Proof.** From (2.5) and the fact that

$$[X, Y] = \nabla_X Y - \nabla_Y X$$

we have

$$\begin{aligned}R(X, Y)\xi &= \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X, Y]}\xi \\ &= \nabla_X\{\delta\beta\varphi^2 Y\} - \nabla_Y\{\delta\beta\varphi^2 X\} \\ &\quad -\delta\beta[\nabla_X Y - \nabla_Y X) + \eta(\nabla_X Y - \nabla_Y X)\xi] \\ &= \delta\{(X\beta)\varphi^2 Y - (Y\beta)\varphi^2 X\} + \beta\delta\{\nabla_X(\varphi^2 Y) - \nabla_Y(\varphi^2 X)\} \\ &\quad - \delta\beta[(\nabla_X Y - \nabla_Y X) + \eta(\nabla_X Y - \nabla_Y X)\xi],\end{aligned} \tag{2.10}$$

Also, we have

(2.11) $$\nabla_X(\varphi^2 Y) = \nabla_X Y + \beta\{g(X, Y) + \delta\eta(X)\eta(Y))\}\xi + \eta(\nabla_X Y)\xi + \delta\beta\eta(Y)\{X + \eta(X)\xi\}$$

From (2.11), finding the expression for

$$\nabla_X(\varphi^2 Y) - \nabla_Y(\varphi^2 X)$$

and further, substituting in (2.10), after simplification, we get (2.8).

(2.9) follows from (2.8) by putting $X = \xi$.

**Lemma 2.4.** *For a δ-Lorentzian β-Kenmotsu manifold M, we have*

(2.12) $$R(\xi, Y)X = \beta^2\{\delta g(X, Y)\xi - \eta(X)Y\} + \delta\{(X\beta)\varphi^2 Y - g(\varphi(X), \varphi(Y))(grad\beta)\}$$

*for any vector fields X and Y on M.*

**Proof.** From the identity

$$g(R(\xi, Y)X, Z) = g(R(X, Z)\xi, Y)$$

and (2.8) of Lemma 2.3, we have

$$\begin{aligned} g(R(\xi, Y)X, Z) &= g(R(X, Z)\xi, Y) \\ &= \beta^2\{-\eta(X)g(Z, Y) + \eta(Z)g(X, Y)\} \\ &+ \delta\{-(Z\beta)g(\varphi^2 X, Y) + (X\beta)g(Z, \varphi^2 Y)\} \end{aligned}$$

After simplification, we (2.12).

**Lemma 2.5.** *For a δ-Lorentzian β-Kenmotsu manifold M, we have*

(2.13) $$S(Y, \xi) = 2n\beta^2\eta(Y) - (2n - 1)\delta(Y\beta) + \delta\eta(Y)(\xi\beta)$$

(2.14) $$S(\xi, \xi) = -2n\{\beta^2 + \delta(\xi\beta)\}$$

(2.15) $$QY = 2n\beta^2 Y,$$

*where* $\beta$ = *constant*

**Proof.** From (2.8), we have

$$(2.16) \qquad g(R(X, Y)\xi, Z) = \beta^2\{\eta(Y)g(X, Z) - \eta(X)g(Y, Z)\} + \delta\{(X\beta)g(\varphi^2 Y, Z) - (Y\beta)g(\varphi^2 X, Z)\}$$

Let $\{e_i\}$, $i = 1, 2, 3..., 2n + 1$ be the orthonormal basis at each point of the tangent space of $M$. Then Putting $X = Z = e_i$, we have

$$g(R(e_i, Y)\xi, e_i) = \beta^2\{\eta(Y)g(e_i, e_i) - \eta(e_i)g(Y, e_i)\} + \delta\{(e_i\beta)g(\varphi^2 Y, e_i) - (Y\beta)g(\varphi^2 e_i, e_i)\}$$

which after simplification gives (2.13). Put $Y = \xi$ in (2.13) to get (2.14). Also from (2.13), we get (2.15).

## 3. GENERALISED RECURRENT δ-LORENTZIAN β-KENMOTSU MANIFOLD

In this Section onwards, we assume that $\beta$ is constant.

**Definition 3.1.** A $\delta$-Lorentzian $\beta$-Kenmotsu manifold $M$ is said to be a generalised recurrent manifold if the curvature tensor $R$ of $M$ satisfies

$$(3.1) \qquad (\nabla_X R)(Y, Z)W = A(X)R(Y, Z)W + B(X)\{g(Z, W)g(Z, W)Y - g(Z, W)Z\},$$

where $A$ and $B$ are associated 1-forms and $X, Y, Z, W$ are any vector fields on $M$.

**Lemma 3.1.** *For a generalised recurrent $\delta$-Lorentzian $\beta$-Kenmotsu manifold $M$, we have*

$$(3.2) \qquad (\nabla_X R)(\xi, Z)\xi = 0$$

*for any vector fields $X$, $Z$ on $M$.*

**Proof.** We know that

$$\begin{aligned}(\nabla_X R)(\xi, Z)\xi &= \nabla_X(R(\xi, Z)\xi - R(\nabla_X\xi, Z)\xi \\ &\quad - R(\xi, \nabla_X Z)\xi - R(\xi, Z)\nabla_X\xi \\ &= \nabla_X[\{\beta^2 + \delta(\xi\beta)\}\varphi^2 Z] \\ &\quad - R[\delta\beta\{X + \eta(X)\xi\}, Z]\xi \\ &\quad - \{\beta^2 + \delta(\xi\beta)\}\varphi^2(\nabla_X Z) \\ &\quad - \delta\beta R(\xi, Z)X - \delta\beta\eta(X)R(\xi, Z)\xi\end{aligned}$$

Now using (2.5), (2.6), (2.8), (2.9) in the above equation, after lengthy simplification, we get (3.2)

We assume that $M$ is a generalised recurrent δ-Lorentzian β-Kenmotsu manifold. Then (3.1) of Definition 3.1 holds. Now put $Y = W = \xi$ in (3.1), we find

(3.3) $$(\nabla_X R)(\xi, Z)\xi = A(X)R(\xi, Z)\xi - B(X)\{g(Z, \xi)\xi - g(\xi, \xi)Z\}$$

By virtue of (3.2) of Lemma 3.1 and the above equation (3.3), we find

$$\beta^2 A(X) + \delta B(X) = 0$$

for any vector field $X$ on $M$. Hence we state

**Theorem 3.1.** *A generalised recurrent δ- Lorentzian β-Kenmotsu manifold M satisfies*

$$\beta^2 A + \delta B = 0$$

*where* $\delta = \pm 1$.

**Corollary 3.1.** *A generalised recurrent Lorentzian β-Kenmotsu manifold M satisfies* $\beta^2 A + B = 0$

**Corollary 3.2.** *A generalised recurrent Lorentzian Kenmotsu manifold M the* 1*-forms A and B are in the opposite direction.*

## 4. RICCI SYMMETRIC AND SEMISYMMETRIC δ-LORENTZIAN β-KENMOTSU MANIFOLD

In this Section, we introduce the notion of Ricci semisymmetric and semi symmetric δ-Lorentzian β-Kenmotsu manifold

**Definition 4.1.** A δ-Lorentzian β-Kenmotsu manifold is said to be Ricci semisymmetric if

(4.1) $$(R(X, Y)\cdot S)(Z, U) = 0$$

and is said to be semisymmetric if

(4.2) $$(R(X, Y)\cdot R)(Z, U) = 0$$

for any vector fields $X, Y, Z, U$ on $M$.

We have

(4.3) $(R(X, Y)\cdot S)(Z, U) = R(X, Y)S(Z, U) - S(R(X, Y)Z, U) - S(Z, R(X, Y)U)$

From (4.1) the above equation (4.3) reduces to

(4.4) $S(R(X, Y)Z, U) + S(Z, R(X, Y)U) = 0$

Now setting $X = Z = \xi$ in (4.4), we get

(4.5) $S(R(\xi, Y)\xi, U) + S(\xi, R(\xi, Y)U) = 0$

Now using (2.12) of Lemma 2.4 and (2.9) of Lemma 2.3 in (4.5), we get

(4.6) $S(Y, U) = (-2n\beta^2\delta)g(Y, U)$

(4.7) $r = -2n(2n + 1)\beta^2\delta$

where $r$ is the scalar curvature of $M$. Hence, we state

**Theorem 4.1.** *A Ricci symmetric $\delta$-Lorentzian $\beta$-Kenmotsu manifold is an Einstein manifold*

**Corollary 4.1.** [9] *A Ricci symmetric Lorentzian $\beta$-Kenmotsu manifold is an Einstein manifold*

**Corollary 4.2.** [9] *A Ricci symmetric Lorentzian Kenmotsu manifold is an Einstein manifold*

**Theorem 4.2.** *A symmetric $\delta$-Lorentzian $\beta$-Kenmotsu manifold is an Einstein manifold*

**Proof.** Follows from the fact that R·R = 0 is the subset of R·S = 0, so that R·S = 0 implies that R·R = 0. Hence (4.4) holds.

**Corollary 4.3.** [9] *A symmetric Lorentzian $\beta$-Kenmotsu manifold is an Einstein manifold*

**Theorem 4.3.** *For a Ricci semisymmetric $\delta$-Lorentzian $\beta$-Kenmotsu manifold $M$, the scalar curvature $r$ of $M$ is constant and is given by (4.7)*

## 5. A $\delta$-LORENTZIAN $\beta$-KENMOTSU MANIFOLD WITH C=0

The Weyl's conformal curvature tensor $C$ of type (1, 3) on $M$ is defined by (5.1)

$$C(X, Y)Z = R(X, Y)Z + \frac{1}{2n+1}[S(X, Z)Y - S(Y, Z)X + g(X, Z)QY - g(X, Z)QX]$$

$$+ \frac{1}{2n(2n+1)}\{g(X, Z)Y - g(Y, Z)X\},$$

where $S(X, Y) = g(QX, Y)$ and $X, Y, Z$ are any vector fields on $M$.

For $n > 1$ it is well known that $M$ is conformally flat if $C$ is identically vanishes on $M$.

**Theorem 5.1.** *A conformally flat δ-Lorentzian β-Kenmotsu manifolds $M(n > 1)$ is an η-Einstein manifold.*

**Proof.** Suppose $M$ is conformally flat. Then $C \equiv 0$, so that (5.1) takes the form

$$(5.2) \qquad R(X, Y)Z = -\frac{1}{2n+1}[S(X, Z)Y - S(Y, Z)X + g(X, Z)QY - g(X, Z)QX]$$

$$-\frac{1}{2n(2n+1)}\{g(X, Z)Y - g(Y, Z)X\},$$

Set $Z = \xi$ in (5.2) and then using (2.13) of Lemma 2.5 and (2.8) of Lemma 2.3, one obtains

$$\begin{aligned}\eta(X)QY - \eta(Y)QX = {} & 2n\beta^2\{\eta(Y)X - \eta(X)Y\} \\ & - (2n - 1)\beta^2\{\eta(Y)X - \eta(X)Y\} \\ & -\frac{\delta r}{2n}\{-\eta(X)Y + \eta(Y)X\}\end{aligned}$$

which after simplification gives

$$(5.3) \qquad \eta(X)QY - \eta(Y)QX = \left\{\beta^2 - \frac{\delta r}{2n}\right\}\{\eta(Y)X - \eta(X)Y\}$$

Putting $Y = \xi$ in (5.3) and using (2.15) of Lemma 2.5, we have

$$QX = \left\{\frac{\delta r}{2n} - \beta^2\right\}X - \left\{(2n-1)\beta^2 - \frac{\delta r}{2n}\right\}\eta(X)\xi.$$

From which, we have

$$(5.4) \qquad S(X, Y) = \left\{\frac{\delta r}{2n} - \beta^2\right\}g(X,Y) - \left\{(2n-1)\beta^2 - \frac{\delta r}{2n}\right\}\eta(X)\eta(Y),$$

which proves the Theorem.

Contracting (5.4), we have the following expression for the scalar curvature $r$ of $M$

$$r = \frac{[2n(2n+1)\delta - 1]\beta^2}{2n-(2n+1)\delta+1} \tag{5.5}$$

provided $\delta \neq 1$..If $\delta = -1$, then (5.5), we have

$$r = \frac{2n(n+1)\beta^2}{2n+1} \tag{5.6}$$

**Theorem 5.2.** *In a conformaly flat δ-Lorentzian β-Kenmotsu manifolds $M$ $(n > 1)$, the scalar curvature $r$ of $M$ is given by* (5.5)

**Theorem 5.3.** *In a conformally flat δ-Lorentzian β-Kenmotsu manifolds $M$ $(n > 1)$, the scalar curvature $r$ of $M$ is constant and given by* (5.6).

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# CHARACTERIZATIONS OF TOPOLOGICAL PROPERTIES VIA STRONG QUASI-UNIFORM COVERS

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**ABSTRACT** : Pervin [6] demonstrated that every topological space is quasi-uniformizable. It has been observed by Brümmer [1] that unlike the case of uniform covers for a uniform space, a quasi-uniform space cannot be characterized by quasi-uniform covers. In [5], the last two authors introduced the idea of strong quasi-uniform covers, and a characterization of a quasi-uniform space in terms of such covers was proved; moreover; a few topological properties were also formulated in this connection. The purpose of this paper is to continue the study and to characterize a few more topological properties in terms of strong quasi-uniform covers. Furthermore, we study the inter connections among the three—quasi-uniformity, topology and strong quasi-uniform cover, in terms of category.

**Keywords** : Quasi-uniformity, transitive base (subbase), strong quasi-uniform cover.

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## 1. INTRODUCTION

In [6] Pervin showed that the development of uniform spaces is a natural one from topological spaces through quasi-uniform spaces. In the same paper he mentioned Császár's [2] assertion that every topological space can be derived from a quasi-uniform space i.e., every topological space is a quasi-uniform space. Now it has been shown [8] that a uniform space can be completely characterized by means of uniform covers. Again, the notion of quasi-uniform covers has been introduced in [3], which is analogous to that of uniform covers. Then intuitively it seems that a quasi-uniform space can be characterized completely by means of quasi-uniform covers. But unfortunately this is not the case. Actually Brümmer pointed this out in [1]. Then in [5], the idea of strong quasi-uniform cover was introduced as a suitable modification of quasi-uniform cover, and quasi-uniform spaces were characterized by such covers. In [5], some topological properties were also characterized by this new type of covers and many other

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topological properties were left out. In this paper some of these remaining topological properties are characterized. Lastly we present quasi-uniformity, topology and strong quasi-uniform cover in terms of category so as to have a better understanding of their interrelations.

## 2. PREREQUISITES

The concept of quasi-uniformity on a non-empty set was studied in [2] and [3].

**Definition 2.1** ([3]). *For a non-empty set $X$, a subcollection $\mathcal{Q}$ of the power set* $\exp(X \times X)$ *of $X \times X$ is called a quasi-uniformity on $X$ if*

1. $\Delta \subseteq Q$, $\forall Q \in \mathcal{Q}$, *where* $\Delta = \{(x, x) : x \in X\}$.
2. $Q \in \mathcal{Q}$ *and* $Q \subseteq P \subseteq X \times X$ *together imply* $P \in \mathcal{Q}$.
3. *for any two members $Q$ and $P$ of $\mathcal{Q}$,* $\exists R \in \mathcal{Q}$ *such that* $R \subseteq Q \cap P$.
4. *for any member $Q$ of $\mathcal{Q}$,* $\exists P \in \mathcal{Q}$ *such that* $P \circ P \subseteq Q$, *where* $R \circ S = \{(x, z) \in X \times X : \exists y \in X$ *such that* $(x, y) \in R$ *and* $(y, z) \in S\}$, *for* $R, S \in \mathcal{Q}$.

*The members of $\mathcal{Q}$ are called entourages of the quasi-uniform space $(X, \mathcal{Q})$.*

**Definition 2.2** ([3]). *On a non-empty set $X$, a subcollection $\mathcal{B}$ of a quasi-uniformity $\mathcal{Q}$ is said to from a base for $\mathcal{Q}$ is for each $Q \in \mathcal{Q}$, $\exists B \in \mathcal{B}$ such that $B \subseteq Q$.*

*Here the quasi-uniformity $\mathcal{Q}$ is said to be the quasi-niformity generated by* $\mathcal{B}$.

**Definition 2.3** ([3]). *Let $(X, \mathcal{Q})$ be a quasi-uniform space. Then $\mathcal{S} \subseteq \mathcal{Q}$ is called a subbase for $\mathcal{Q}$ if the family of finite intersections of members of $\mathcal{S}$ is a base for $\mathcal{Q}$.*

*Here the quasi-uniformity $\mathcal{Q}$ is said to be the quasi-uniformity generated by* $\mathcal{S}$.

**Theorem 2.4** ([3]). *Let $(X, \mathcal{Q})$ be a quasi-uniform space. Then the collection* $\{G \subseteq X : x \in G \Rightarrow x \in Q(x) \subseteq G$, *for some* $Q \in \mathcal{Q}\}$ *forms a topology on $X$, where* $Q(x) = \{y \in X : (x, y) \in Q\}$.

*This topology is termed as the topology induced by $\mathcal{Q}$ on $X$ and it is denoted by* $\tau(\mathcal{Q})$.

In [2] Császár asserted that every topological space can be derived from a quasi-uniform space. This quasi-uniformity for a given topological space is called a compatible quasi-

uniformity with the topology, and the given topology is said to admit the quasi-uniformity. In [7], Pervin gave a direct topological construction of a compatible quasi-uniformity for a given topological space and it is called the Pervin quasi-uniformity for the space.

**Definition 2.5** ([3]). *A base (subbase) for a quasi-uniformity is said to be transitive if $B \circ B = B$, for all members $B$ of the base (subbase), so that each member of the base (subbase) is a transitive relation.*

**Definition 2.6** ([3]). *If a quasi-uniformity has a transitive base (or subbase), then it is said to be a transitive quasi-uniformity.*

**Example 2.7** ([3]). *Let $(X, \tau)$ be a topological space. Pervin [7] constructed a quasi-uniformity on $(X, \tau)$ with the sbbase given by th collection, $\{T(G, X \setminus G) : G \in \tau\}$, where $T(G, X \setminus G)$ stands for $(X \times X \setminus (G \times (X \setminus G)))$. Then the collection of the sets of the form*

$$\bigcap_{i\in\{1,2,\ldots,n\}} T(G_i, X\setminus G_i) = \bigcap_{\Lambda\subseteq\{1,2,\ldots,n\}} \Big(\Big(\bigcap_{i\in\Lambda} G_i \setminus \bigcup_{j\in\{1,2,\ldots,n\}\setminus\Lambda} G_j\Big) \times \bigcap_{i\in\Lambda} G_i\Big) \cup \Big(\Big(X \setminus \bigcup_{i\in\{1,2,\ldots,n\}} G_i\Big) \times X\Big)$$

*turn out to be a base for this quasi-uniformity, $\forall n \in \mathbb{N}$. It is called Pervin's quasi-uniformity. For a topological space, the Pervin's quasi-uniformity is transitive.*

**Definition 2.8** ([3]). *For two quasi-uniform spaces $(X, \mathcal{Q}_X)$ and $(Y, \mathcal{Q}_Y)$, a function $f : X \to Y$ is called quasi-uniformly continuous if for each $Q \in \mathcal{Q}_Y$, $\exists P \in \mathcal{Q}_X$ such that $(x_1, x_2) \in P \Rightarrow (f(x_1), f(x_2)) \in Q$.*

## 3. COVERS FOR QUASI-UNIFORM SPACES

We start with the existing notion of quasi-uniform cover for a quasi-uniform space, which is analogous to that of a uniform cover for a uniform space.

**Definition 3.1** ([3]). *A cover $\mathcal{C}$ of a subset $A$ of a quasi-uniform space $(X, \mathcal{Q})$ is said to be a quasi-uniform cover of $A$ if $\exists Q \in \mathcal{Q}$ such that for each $a \in A$, $\exists C \in \mathcal{C}$ with $a \in Q(a) \subseteq C$.*

**Example 3.2.** *Let $(X, \mathcal{Q})$ be a quasi-uniform space and $\tau = \tau(\mathcal{Q})$ be the topology induced by $\mathcal{Q}$ on $X$. Then $\tau$ itself is a quasi-uniform cover of $X$.*

Now, unlike uniform cover for a uniform space, quasi-uniform cover could not describe quasi-uniformity completely (see [1] for further details). Then in [5], the corresponding authors

suitably amended the notion of quasi-uniform cover to have the new one, called strong quasi-uniform cover. In this section first we recall the basics about strong quasi-uniform cover from [5] and then interpret certain characteristics of it to have all the prerequisites for our further discussion. Before proceeding further, we fix here certain notations which will be used afterwards. For a non-empty set $X$ and a non-empty collection $\mathcal{C} \subseteq \exp(X)$, the collection $\{C \in \mathcal{C} : x \in C\}$ will be denoted by $\mathcal{C}_x$, for each $x \in X$. Further by $Q_{\mathcal{C}}$ we will mean the collection $\{(x, y) : x \in X \text{ and } y \in \cap\, \mathcal{C}_x\}$. In [3], $Q_{\mathcal{C}}$ was shown to be a reflexive and transitive relation.

**Definition 3.3** ([5]). *Let $(X, \mathcal{Q})$ be a quasi-uniform space and $\mathcal{C}$ be a cover of $X$. Then $\mathcal{C}$ is said to be a strong quasi-uniform cover of $X$ if $x \in Q(x) \subseteq \cap\, \mathcal{C}_x$, for some $Q \in \mathcal{Q}$ and $\forall x \in X$.*

Now for a quasi-uniform space $(X, \mathcal{Q})$, we immediately have a strong quasi-uniform cover, as the singletone collection $\{X\}$. But it's a trivial one. For non-trivial ones we refer to the following lemma which actually gives us a plenty of strong quasi-uniform covers on a given quasi-uniform space.

**Lemma 3.4** ([5]). *For each transitive member $Q$ of a quasi-uniformity $\mathcal{Q}$ on a non-empty set $X$, $\{Q(x) : x \in X\}$ forms a strong quasi-uniform cover of $X$.*

It is quite obvious that a strong quasi-uniform cover is a quasi-uniform cover, but not conversely, which follows immediately from Example 3.2.

**Theorem 3.5.** *Let $\mathcal{B}$ be a transitive base for a compatible quasi-uniformity of a topological space $(X, \tau)$. Then $\{B(x) : x \in X, B \in \mathcal{B}\}$ forms an open base for $\tau$.*

**Proof.** In [3] it has been established that $\{B(x) : B \in \mathcal{B}\}$ is a base for the neighbourhood filter at $x$, $\forall x \in X$. So we only have to show that $B(x) \in \tau$, $\forall x \in X$ and $\forall B \in \mathcal{B}$. Let $y \in B(x)$ and $z \in B(y)$. So $(x, y), (y, z) \in B \Rightarrow (x, z) \in B \circ B = B \Rightarrow z \in B(x) \Rightarrow B(y) \subseteq B(x)$ i.e., $y \in B(y) \subseteq B(x)$, $\forall y \in B(x)$. Hence $B(x) \in \tau$. The rest of the proof follows immediately.

**Theorem 3.6.** *Each strong quasi-uniform cover of a quasi-uniform space is an open cover.*

**Proof.** It follows from the definitions of the strong quasi-uniform cover and the topology generated by a quasi-uniformity.

In [5], the following characterization of a quasi-uniform space was obtained in terms of covers :

**Theorem 3.7.** *Let $\mathscr{C}$ be the collection of all strong quasi-uniform covrs of a quasi-uniform space* $(X, \mathcal{Q})$. Then

1. *If* $\mathsf{C} \in \mathscr{C}$ *and* $\mathsf{D}$ *is a cover of* $X$ *such that* $\cap \mathsf{C}_x \subseteq \cap \mathsf{D}_x$ *for each* $x \in X$, *then* $\mathsf{D} \in \mathscr{C}$.

2. *If* $\mathsf{C}_1, \mathsf{C}_2 \in \mathscr{C}$, *then* $\exists \mathsf{C} \in \mathscr{C}$ *such that* $\cap \mathsf{C}_x \subseteq (\mathsf{C}_1)_x$ *and* $\cap \mathsf{C}_x \subseteq \cap(\mathsf{C}_2)_x$ *for each* $x \in X$.

*Conversely, let $\mathscr{C}$ be a collection of covers of a non-empty set $X$ and $\mathscr{C}$ satisfies* (1) *and* (2). *Then* $\{Q_{\mathsf{C}} : \mathsf{C} \in \mathscr{C}\}$ *forms a transitive base for a quasi-uniformity on $X$ and $\mathscr{C}$ is exactly the collection of all strong quasi-uniform covers of this quasi-uniform space.*

**Theorem 3.8.** *Let $\mathcal{Q}$ be a compatible quasi-uniformity for a topological space $(X, \tau)$ and $\mathscr{C}$ be the collection of all strong quasi-uniform covers of $X$. Then the transitive quasi-uniformity, say $\mathcal{Q}_{\mathscr{C}}$, induced by $\mathscr{C}$ on $X$ is a subcollection of $\mathcal{Q}$ and if $\mathcal{Q}$ is transitive then both are same.*

**Proof.** Let $\mathsf{C} \in \mathscr{C}$. Then for some $Q \in \mathcal{Q}$, $Q(x) \subseteq \cap \mathsf{C}_x$, $\forall x \in X$. Now $Q \subseteq Q_{\mathsf{C}}$. In fact, $(x, y) \in Q \Rightarrow y \in Q(x) \subseteq \cap \mathsf{C}_x \Rightarrow (x, y) \in Q_{\mathsf{C}}$. Thus $Q_{\mathsf{C}} \in \mathcal{Q}$ and hence $\mathcal{Q}_{\mathscr{C}} \subseteq \mathcal{Q}$.

Now let $\mathcal{B}$ be a transitive base for $\mathcal{Q}$, and consider any $B \in \mathcal{B}$. Then by Lemma 3.4, $\mathsf{C} = \{B(x) : x \in X\} \in \mathscr{C}$. We claim that $Q_{\mathsf{C}} \subseteq B$ so that $\mathcal{Q} \subseteq \mathcal{Q}_{\mathscr{C}}$ and we are done. In fact, $(x, y) \in Q_{\mathsf{C}} \Rightarrow y \in \mathsf{C}_x = \cap\{B(z) : x \in B(z), z \in X\} \subseteq B(x)$, as $x \in B(x)$. $\Rightarrow (x, y) \in B$.

From the above result we can infer that distinct quasi-uniformities on a non-empty set may have exactly the same collection of strong quasi-uniform covers.

**Theorem 3.9.** *Let $(X, \tau)$ be a topological space and $\mathcal{Q}$ be a compatible transitive quasi-uniformity for it. The* $\{Q_{\mathsf{C}}(x) : x \in X, \mathsf{C} \in \mathscr{C}\}$ *i.e.,* $\{\cap \mathsf{C}_x : x \in X, \mathsf{C} \in \mathscr{C}\}$ *is a base for* $(X, \tau)$, *where $\mathscr{C}$ is the collection of all strong quasi-uniform covers of* $(X, \mathcal{Q})$.

**Proof.** Follows from Theorem 3.5, 3.7 and 3.8 together.

**Theorem 3.10.** *Let $(X, \mathcal{Q}_X)$ and $(Y, \mathcal{Q}_Y)$ be two quasi-uniform spaces and suppose $\mathcal{Q}_Y$ has a transitive base, say $\mathcal{B}$. Then a function $f : X \to Y$ is quasi-uniformly continuous if and only if given any strong quasi-uniform cover $\mathcal{C}$ of $Y$, $f^{-1}(\mathcal{C}) = \{f^{-1}(C) : C \in \mathcal{C}\}$ is a strong quasi-uniform cover of $X$.*

**Proof.** Let $f: (X, \mathcal{Q}_X) \to (Y, \mathcal{Q}_Y)$ be quasi-uniformly continuous and $\mathcal{C}$ be a strong quasi-uniform cover of $Y$. Then $\mathcal{C}$ being a strong quasi-uniform cover of $Y$, $\exists Q \in \mathcal{Q}_Y$ such that $y \in Q(y) \subseteq \cap \mathcal{C}_y$, $\forall y \in Y$. Now $f$ being quasi-uniformly continuous, $\exists P \in \mathcal{Q}_X$ such that $(x_1, x_2) \in P \Rightarrow (f(x_1), f(x_2)) \in Q$. Next let $x \in X$ and $x' \in P(x)$. Also consider $C \in \mathcal{C}$ such that $x \in f^{-1}(C)$. Then $f(x) \in C \Rightarrow f(x) \in Q(f(x)) \subseteq C$. Again $x' \in P(x) \Rightarrow (x, x') \in P \Rightarrow (f(x), f(x')) \in Q \Rightarrow f(x') \in Q(f(x)) \subseteq C$. Since $C \in \mathcal{C}$ is arbitrary, $f(x') \in Q(f(x)) \subseteq \cap \mathcal{C}_{f(x)}$. So, $P(x) \subseteq \cap(f^{-1}(\mathcal{C}))_x$. Thus $f^{-1}(\mathcal{C})$ is a strong quasi-uniform cover of $X$.

Conversely suppose that the given condition holds and let $Q$ be a member of $\mathcal{Q}_Y$. Then $\mathcal{B}$ being a base for $\mathcal{Q}_Y$, $\exists B \in \mathcal{B}$ such that $B \subseteq Q$. Again by the Theorem 3.4, $\{B(y) : y \in Y\}$ is a strong quasi-uniform cover of $Y$. Now arguing similarly as in Theorem 3.5, we can show that $y \in B(y) \subseteq \cap\{B(z) : y \in B(z) \text{ and } z \in Y\}$, $\forall y \in Y$. Now by the assumed condition, $\{f^{-1}(B(y)) : y \in Y\}$ is a strong quasi-uniform cover of $X$. So, there exist $P \in \mathcal{Q}_X$ such that $x \in P(x) \subseteq \cap\{f^{-1}(B(y)) : x \in f^{-1}(B(y)) \text{ and } y \in Y\}$, $\forall x \in X$. Let $(x_1, x_2) \in P$. Then $x_2 \in P(x_1) \subseteq \cap\{f^{-1}(B(y)) : x_1 \in f^{-1}(B(y)) \text{ and } y \in Y\}$. Again $\cap\{f^{-1}(B(y)) : x_1 \in f^{-1}(B(y)) \text{ and } y \in Y\} \subseteq f^{-1}(B(f(x_1)))$. So $x_2 \in f^{-1}(B(f(x_1))) \Rightarrow f(x_2) \in B(f(x_1)) \Rightarrow (f(x_1), f(x_2)) \in B$. Thus $f$ is quasi-uniformly continuous.

The transitivity of $(Y, \mathcal{Q}_Y)$ is not necessary for the necessity part.

## 4. TOPOLOGICAL PROPERTIES AND STRONG QUASI-UNIFORM COVERS

Here in this section we will characterize some topological properties in terms of strong quasi-uniform covers. In [5], some properties, namely Hausdorffness, compactness, near compactness, paracompactness, near paracompactness, S-closedness, s-closedness, quasi-H-closedness, have been characterized. Here we will address the issues regarding first countability, several separation axioms, separability and connectedness. To that end, we first derive the expressions of closure and intrior of a subset in terms of strong quasi-uniform covers of a quasi-uniform space.

In what follows in this section, it will be assumed (if not stated otherwise), to avoid repetition, that the topology $\tau$ of any space $(X, \tau)$, under consideration, is generated by a transitive quasi-uniform base $\mathcal{B}$ (in fact, this is always the case in view of Example 2.7).

**Lemma 4.1.** *Suppose $\mathscr{C}$ is the collection of all strong quasi-uniform covers of a topological space $(X, \tau)$. Then for any subset $A$ of $X$,*

$$cl(A) = \{x \in X : (\cap \mathrm{C}_x) \cap A \neq \phi,\ \forall \mathrm{C} \in \mathscr{C}\}.$$

**Proof.** Let $x \in cl(A)$ and $\mathrm{C} \in \mathscr{C}$. As $x \in \cap\mathrm{C}_x \in \tau$, we have $(\cap\mathrm{C}_x) \cap A \neq \phi$.

Again, let $x \in X$ such that $(\cap\mathrm{C}_x) \cap A \neq \phi$, $\forall \mathrm{C} \in \mathscr{C}$. Then for any open neighbourhood $G$ of $x$, $\exists \mathrm{C} \in \mathscr{C}$ such that $x \in \cap\, \mathrm{C}_x \subseteq G$. So $A \cap G \neq \phi$ and hence $x \in cl(A)$.

**Lemma 4.2.** *If $\mathscr{C}$ denotes the family of all strong quasi-uniform covers of a topological space $(X, \tau)$, then for any subset $A$ of $X$,*

$$int(A) = \{x \in X : \cap\mathrm{C}_x \subseteq A, \text{ for some } \mathrm{C} \in \mathscr{C}\}.$$

**Proof.** Let $x \in int(A)$. Then the equality follows immediately by virtue of Theorem 3.9.

Now, let $x \in X$ with $\cap\mathrm{C}_x \subseteq A$ for some $\mathrm{C} \in \mathscr{C}$. Now as $\mathrm{C} \in \mathscr{C}$, $\exists B \in \mathcal{B}$ such that $x \in B(x) \subseteq \cap\mathrm{C}_x$ i.e., $x \in B(x) \subseteq A$. Again $B(x) \in \tau$. Thus $x \in int(A)$.

We are now equipped enough to characterize certain topological concepts via strong quasi-uniform covers.

**Theorem 4.3.** *Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be two topological spaces, where $\tau_X$ and $\tau_Y$ are generated by two quasi-uniformities with transitive bases $\mathcal{B}_X$ and $\mathcal{B}_Y$ respectively. Then a function $f : X \to Y$ is continuous if and only if given $x \in X$ and a strong quasi-uniform cover $\mathrm{C}_Y$ of $Y$, $\exists$ a strong quasi-uniform cover $\mathrm{C}_X$ of $X$ such that $f(\cap(\mathrm{C}_X)_x) \subseteq \cap(\mathrm{C}_Y)_{f(x)}$.*

**Proof.** Let $f : X \to Y$ be continuous and $x \in X$. Again let $\mathrm{C}_Y$ be a strong quasi-uniform cover of $Y$. Then, by Theorem 3.9, $\cap(\mathrm{C}_Y)_{f(x)}$ is an open neighbourhood of $f(x)$. Now by using the same theorem and the continuity of $f$, we find a strong quasi-uniform cover $\mathrm{C}_X$ of $X$ such that $x \in \cap(\mathrm{C}_X)_x$ and $f(\cap(\mathrm{C}_X)_x) \subseteq \cap(\mathrm{C}_Y)_{f(x)}$.

Conversely we assume the given condition and let $x \in X$ and $f(x) \in G \in \tau_Y$. Then $f(x) \in \cap(\mathrm{C}_Y)_{f(x)} \subseteq G$, for some strong quasi-uniform cover $\mathrm{C}_Y$ of $Y$. By hypothesis, there exists a

strong quasi-uniform cover $\mathcal{C}_Y$ of $X$ such that $f(\cap(\mathcal{C}_X)_x) \subseteq \cap(\mathcal{C}_Y)_{f(x)}$, where $x \in \cap(\mathcal{C}_X)_x \in \tau_X$, proving the continuity of $f$.

**Theorem 4.4.** *A topological space $(X, \tau)$ is $T_0$ if and only if for each pair of distinct points $x$ and $y$ of $X$, there exists a strong quasi-uniform cover $\mathcal{C}$ of $X$ such that either $y \notin \cap\mathcal{C}_x$ or $x \notin \cap\mathcal{C}_y$.*

**Proof.** Let $x, y$ be two distinct points in a $T_0$-space $(X, \tau)$. Then for some open set $G$ of $X$, either $x \in G, y \notin G$ or $x \notin G, y \in G$. Then for the first case, there exists a strong quasi-uniform cover $\mathcal{C}$ of $X$ such that $x \in \cap\mathcal{C}_x \subseteq G, y \notin \cap\mathcal{C}_x$. Similar is the other case. Thus the result follows.

The converse is also clear in view of Theorem 3.9.

**Theorem 4.5.** *A topological space $(X, \tau)$ is $T_1$ if and only if for each pair of distinct points $x$ and $y$ of $X$, there exists a strong quasi-univorm cover $\mathcal{C}$ of $X$ such that $y \notin \cap\mathcal{C}_x$ and $x \notin \cap\ \mathcal{C}_y$.*

**Proof.** Let $x, y$ be two distinct points in a $T_1$-space $(X, \tau)$. Then there are open sets $G_x, G_y$ in $X$ such that $x \in G_x, y \notin G_x$ and $y \in G_y, x \notin G_y$. Then there exist strong quasi-uniform covers $\mathcal{C}_1, \mathcal{C}_2$ of $X$ such that $x \in \cap\ (\mathcal{C}_1)_x \subseteq G_x$ and $y \in \cap\ (\mathcal{C}_2)_y \subseteq G_y$. Now by Theorem 3.7, there exists a strong quasi-uniform cover $\mathcal{C}$ of $X$ such that $x \in \cap\ \mathcal{C}_x \subseteq \cap(\mathcal{C}_1)_x$ and $y \in \cap\ \mathcal{C}_y \subseteq \cap\ (\mathcal{C}_2)_y$. This gives $y \notin \cap\ \mathcal{C}_x$ and $x \notin \cap\ \mathcal{C}_y$.

The converse follows from the fact that $\cap\mathcal{C}_x, \cap\mathcal{C}_y \in \tau$.

**Theorem 4.6.** *A topological space $(X, \tau)$ is $T_2$ if and only if for any two distinct points $x$ and $y$ of $X$, there exists a strong quasi-uniform cover $\mathcal{C}$ of $X$ such that $(\cap\mathcal{C}_x)\cap(\cap\mathcal{C}_y) = \phi$.*

**Proof.** Consider any two distinct points $x, y$ in a $T_2$-space $(X, \tau)$. Then there are two disjoint open sets $U, V$ in $X$ such that $x \in U, y \in V$ and $U \cap V = \phi$. Then we have, $x \in \cap(\mathcal{C}_1)_x \subseteq U$ and $y \in \cap\ (\mathcal{C}_2)_y \subseteq V$, for some strong quasi-uniform covers $\mathcal{C}_1, \mathcal{C}_2$ of $X$. Now by Theorem 3.7, there exists a strong quasi-uniform cover $\mathcal{C}$ of $X$ such that $x \in \cap\mathcal{C}_x \subseteq \cap(\mathcal{C}_1)_x$ and $y \in \cap\mathcal{C}_y \subseteq \cap(\mathcal{C}_2)_y$. Since $U \cap V = \phi$, we have $(\cap\mathcal{C}_x) \cap (\cap\mathcal{C}_y) = \phi$.

Conversely since $\cap\mathcal{C}_x, \cap\mathcal{C}_y \in \tau(\mathcal{B}) = \tau$, $X$ becomes $T_2$.

**Note 4.7.** *Though the above theorem was proved in [5], here we have given a simpler proof.*

The proofs of the next two theorems go along the same line as those of the above three theorems and hence are omitted.

**Theorem 4.8.** *A topological space $(X, \tau)$ is regular if and only if for any given closed set $A \subseteq X$ and $x \in X \setminus A$, there exists a strong quasi-uniform cover $\mathcal{C}^y$ of $X$, for each $y \in A \cup \{x\}$ such that $(\cap \mathcal{C}^x{}_x) \cap (\cap \mathcal{C}^a{}_a) = \phi$, $\forall a \in A$.*

**Theorem 4.9.** *A topological space $(X, \tau)$ is normal if and only if for any two non-empty closed sets $A$, $D \subseteq X$ with $A \cap D = \phi$, there exist strong quasi-uniform covers $\mathcal{C}^x$ of $X$, for each $x \in A \cup D$, such that $(\cap \mathcal{C}^a{}_a) \cap (\cap \mathcal{C}^d{}_d) = \phi$, $\forall a \in A$ and $\forall d \in D$.*

**Theorem 4.10.** *A topological space $(X, \tau)$ is connected if and only if for any non-empty proper subset $A$ of $X$ and any strong quasi-uniform cover $\mathcal{C}$ of $X$, $\exists x \in X$ such that $A \cap (\cap \mathcal{C}_x) \neq \phi \neq (X \setminus A) \cap (\cap \mathcal{C}_x)$.*

**Proof.** Let $(X, \tau)$ be connected and $A$ be a non-empty proper subset of $X$. As $Bd(A)$ (=boundary of $A$) $\neq \phi$, choose $x \in Bd(A)$. Then by Theorem 3.9, Lemma 4.1 and Lemma 4.2, the condition follows immediately.

Conversely suppose that $A$ is a non-empty proper subset of $X$ and the given condition holds for $A$. then again by Theorem 3.9, Lemma 4.1 and Lemma 4.2, $x \in Bd(A)$ i.e., $A$ has non-empty boundary and so $(X, \tau)$ is connected.

**Theorem 4.11.** *A topological space $(X, \tau)$ is first countable if and only if given $x \in X$, $\exists$ a countable family $\{\mathcal{C}_n : n \in \mathbb{N}\}$ of strong quasi-uniform covers of $X$ such that $\{\cap(\mathcal{C}_n)_x : n \in \mathbb{N}\}$ is a local base at $x$.*

**Proof.** Let in a first countable topological space $(X, \tau)$, $\{A_n : n \in \mathbb{N}\}$ be a local base at $x \in X$. As $A_n \in \tau$ for each $n \in \mathbb{N}$, there exists a strong quasi-uniform cover $\mathcal{C}_n$ of $X$ such that $x \in \cap(\mathcal{C}_n)_x \subseteq A_n$, proving that $\{\cap(\mathcal{C}_n)_x : n \in \mathbb{N}\}$ is a local base at $x$.

The converse part is immediate by use of Theorem 3.9.

**Theorem 4.12.** *A topological space $(X, \tau)$ is separable if and only if there exist a countable subset $A$ of $X$ such that for any strong quasi-uniform cover $\mathcal{C}$ of $X$.*

$$(\cap \mathcal{D}) \cap A \neq \phi, \ \forall \mathcal{D} \subseteq \mathcal{C} \text{ with } \cap \mathcal{D} \neq \phi.$$

**Proof.** Let $(X, \tau)$ be separable. Then for some countable subset $A$ of $X$, $cl(A) = X$. Now consider a strong quasi-uniform cover $\mathsf{C}$ of $X$ and suppose $\mathsf{D} \subseteq \mathsf{C}$ with $\cap\mathsf{D} \neq \phi$. Also, let $x \in \cap\mathsf{D}$. Now as $x \in cl(A)$, $(\cap\mathsf{C}_x) \cap A \neq \phi$, by Lemma 4.1. Since $\cap\mathsf{C}_x \subseteq \cap\mathsf{D}$, we have $(\cap\mathsf{D}) \cap A \neq \phi$.

Conversely assuming the given condition, let $x \in X$ and $x \in G \in \tau$. Then $x \in \cap\mathsf{C}_x \subseteq G$, for some strong quasi-uniform cover $\mathsf{C}$ of $X$. We need to show that $A \cap G \neq \phi$, which is immediate from the hypothesis that $(\cap\mathsf{C}_x) \cap A \neq \phi$.

## 5. CATEGORICAL DESCRIPTION OF QUASI-UNIFORMITY

In this section we will describe quasi-uniformity and its interrelations with topology and strong quasi-uniform cover in terms of category. To do this at first we will consider the following definitions and results. All the concepts regarding category used here are taken from [4].

**Definition 5.1.** *Let $X$ be a non-empty set. A covering structure on $X$ is a collection $\mathscr{C}$ of covers of $X$ such that*

1. *$\mathsf{C} \in \mathscr{C}$ and $\mathsf{D}$ is a cover of $X$ such that $\cap\mathsf{C}_x \subseteq \cap\mathsf{D}_x$, for each $x \in X$ implies $\mathsf{D} \in \mathscr{C}$.*
2. *$\mathsf{C}_1, \mathsf{C}_2 \in \mathscr{C}$ implies $\exists\mathsf{C} \in \mathscr{C}$ such that $\cap\mathsf{C}_x \subseteq \cap(\mathsf{C}_1)_x$ and $\cap\mathsf{C}_x \subseteq \cap(\mathsf{C}_2)_x$, for each $x \in X$.*

*$X$ together with $\mathscr{C}$, $(X, \mathscr{C})$, is called a covering structure space.*

**Theorem 5.2.** *Let $(X, \mathcal{Q}_X)$ and $(Y, \mathcal{Q}_Y)$ be two quasi-uniform spaces and $\tau_X$ and $\tau_Y$ be two compatible topologies with $\mathcal{Q}_X$ and $\mathcal{Q}_Y$ respectively. Then the quasi-uniform continuity of $f : (X, \mathcal{Q}_X) \to (Y, \mathcal{Q}_Y)$ implies the continuity of $f : (X, \tau_X) \to (Y, \tau_Y)$.*

**Proof.** The proof is quite straightforward.

**Theorem 5.3.** *Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be two topological spaces and $\mathcal{P}_X$ and $\mathcal{P}_Y$ be the Pervin's quasi-uniformities generated by $\tau_X$ and $\tau_Y$ respectively. Then the continuity of $f : (X, \tau_X) \to (Y, \tau_Y)$ implies the quasi-uniform continuity of $f : (X, \mathcal{P}_X) \to (Y, \mathcal{P}_Y)$.*

**Proof.** The proof is absolutely straightforward, but involves rigorous calculations. Hence we omit it.

**Definition 5.4.** *Let $X$ and $Y$ be two non-empty sets and $\mathscr{C}_X$ and $\mathscr{C}_Y$ be two collections of covers on $X$ and $Y$ respectively. Then a function $f : (X, \mathscr{C}_X) \to (Y, \mathscr{C}_Y)$ is said to be cover continuous if for any $\mathsf{C} \in \mathscr{C}_Y$, $f^{-1}(\mathsf{C}) = \{f^{-1}(C) : C \in \mathsf{C}\}$ is a member of $\mathscr{C}_X$.*

**Theorem 5.5.** *Let $(X, Q_X)$ and $(Y, Q_Y)$ be two quasi-uniform spaces and $\mathscr{C}_X$ and $\mathscr{C}_Y$ be the collections of strong quasi-uniform covers of $X$ and $Y$ respectively. Then the quasi-uniform continuity of $f : (X, Q_X) \to (Y, Q_Y)$ implies the cover continuity of $f : (X, \mathscr{C}_X) \to (Y, \mathscr{C}_Y)$.*

**Proof.** It follows readily from Theorem 3.10.

**Theorem 5.6.** *Let $(X, \mathscr{C}_X)$ and $(Y, \mathscr{C}_Y)$ be two covering structure spaces and $Q_X$ and $Q_Y$ be the quasi-uniformities generated by $\mathscr{C}_X$ and $\mathscr{C}_Y$ respectively. Then the cover continuity of $f : (X, \mathscr{C}_X) \to (Y, \mathscr{C}_Y)$ implies the quasi-uniform continuity of $f : (X, Q_X) \to (Y, Q_Y)$.*

**Proof.** It again follows from Theorem 3.10.

We now list some major facts about continuous, quasi-uniformly continuous and cover continuous functions to construct certain categories afterwards.

**Fact 5.7.** 1. *Composition of any two quasi-uniformly continuous (continuous, cover continuous) functions between two quasi-uniform (topological, covering structure) spaces is quasi-uniformly continuous (continuous, cover continuous).*

2. *The identity function on a quasi-uniform (topological, covering structure) space is quasi-uniformly continuous (continuous, cover continuous).*

In view of the above facts we consider the following categories :

1. *QU* : The elements are the quasi-uniform spaces and the morphisms are the quasi-univormly continuous functions among them.
2. *QU** : The elements are the transitive quasi-uniform spaces and the morphisms are the quasi-uniformly continuous functions among them.
3. *Top* : The elements are the topological spaces and the morphisms are the continuous functions among them.
4. *Cov* : The elements are the covering structure spaces and the morphisms are the cover-continuous functions among them.

**Theorem 5.8.** *$QU^*$ is a full subcategory of $QU$.*

**Proof.** It is obvious.

**Theorem 5.9.** *$F : QU \to Top$, described by*

$$\begin{array}{ccc} (X, Q_X) & \mapsto & (X, \tau_X) \\ f\big\downarrow & & \big\downarrow F(f) = f \\ (Y, Q_Y) & \mapsto & (Y, \tau_Y) \end{array}$$

*is a faithful, but not a full functor, where $\tau_X = \tau(Q_X)$, $\tau_Y = \tau(Q_Y)$.*

**Proof.** The first part follows from Theorem 5.2.

For the last part we consider the real line $\mathbb{R}$. Clearly the uniformity generated by the usual metric $d$ on $\mathbb{R}$ is a quasi-uniformity on $\mathbb{R}$. There are plenty of real-valued continuous functions on $\mathbb{R}$ which are not uniformly continuous. So the functor $F$ cannot be full in general.

**Theorem 5.10.** *$F^* : QU^* \to Top$ described by*

$$\begin{array}{ccc} (X, Q_X) & \mapsto & (X, \tau_X) \\ f\big\downarrow & & \big\downarrow F(f) = f \\ (Y, Q_Y) & \mapsto & (Y, \tau_Y) \end{array}$$

*is a faithful, but not a full functor, where $\tau_X = \tau(Q_X)$, $\tau_Y = \tau(Q_Y)$.*

**Proof.** The proof is same as that of the previous one.

**Theorem 5.11.** *$G : Top \to QU$ described by*

$$\begin{array}{ccc} (X, \tau_X) & \mapsto & (X, \mathcal{P}(\tau_X)) \\ f\big\downarrow & & \big\downarrow F(f) = f \\ (Y, \tau_Y) & \mapsto & (Y, \mathcal{P}(\tau_Y)) \end{array}$$

*is a fully faithful functor, where $\mathcal{P}(\tau_X)$ and $\mathcal{P}(\tau_Y)$ are the Pervin's quasi-uniformities, generated by $\tau_X$ and $\tau_Y$ respectively.*

**Proof.** The proof follows from Theorems 5.2 and 5.3.

**Theorem 5.12.** *$G^*$ : Top $\to$ QU* described by*

$$\begin{array}{ccc} (X,\ \tau_X) & \mapsto & (X,\ \mathcal{P}(\tau_X)) \\ f\big\downarrow & & \big\downarrow F(f) = f \\ (Y,\ \tau_Y) & \mapsto & (Y,\ \mathcal{P}(\tau_Y)) \end{array}$$

*is a fully faithful functor, where $\mathcal{P}(\tau_X)$* and *$\mathcal{P}(\tau_Y)$ are the Pervin's quasi-uniformities, generated by $\tau_X$ and $\tau_Y$ respectively.*

**Proof.** The proof is similar to that of the above Theorem.

**Theorem 5.13.** *$I$ : QU $\to$ Cov described by*

$$\begin{array}{ccc} (X,\ Q_X) & \mapsto & (X,\ \mathscr{C}_X) \\ f\big\downarrow & & \big\downarrow F(f) = f \\ (Y,\ Q_Y) & \mapsto & (Y,\ \mathscr{C}_Y) \end{array}$$

*is a fully faithful functor, where $\mathscr{C}_X$ and $\mathscr{C}_Y$ are the collections of strong quasi-uniform covers of $X$ and $Y$ respectively.*

**Proof.** It follows easily from Theorem 3.10.

**Theorem 5.14.** *$J$ : Cov $\to$ QU described by*

$$\begin{array}{ccc} (X,\ \mathscr{C}_X) & \mapsto & (X,\ Q(\mathscr{C}_X)) \\ f\big\downarrow & & \big\downarrow F(f) = f \\ (Y,\ \mathscr{C}_Y) & \mapsto & (Y,\ Q(\mathscr{C}_Y)) \end{array}$$

*is a fully faithful functor, where $Q(\mathscr{C}_X)$ and $Q(\mathscr{C}_Y)$ are the quasi-uniformities, generated by $\mathscr{C}_X$ and $\mathscr{C}_Y$ respectively.*

**Proof.** Follows from Theorems 3.7 and 3.10.

**Theorem 5.15.** *Let us consider the following two categories :*

*$I^*$ : $QU^* \to Cov$ described by*

$$\begin{tikzcd} (X, Q_X) \arrow[r, mapsto] \arrow[d, "f"'] & (X, \mathscr{C}_X) \arrow[d, "F(f) = f"] \\ (Y, Q_Y) \arrow[r, mapsto] & (Y, \mathscr{C}_Y) \end{tikzcd}$$

*where $\mathscr{C}_X$ and $\mathscr{C}_Y$ are the collections of strong quasi-uniform covers of $X$ and $Y$ respectively.*

*and*

*$J^*$ $Cov \to QU^*$ described by*

$$\begin{tikzcd} (X, \mathscr{C}_X) \arrow[r, mapsto] \arrow[d, "f"'] & (X, Q(\mathscr{C}_X)) \arrow[d, "F(f) = f"] \\ (Y, \mathscr{C}_Y) \arrow[r, mapsto] & (Y, Q(\mathscr{C}_Y)) \end{tikzcd}$$

*where $Q(\mathscr{C}_X)$ and $Q(\mathscr{C}_Y)$ are the quasi-uniformities, generated by $\mathscr{C}_X$ and $\mathscr{C}_Y$ respectively.*

*Then both $I^*$ and $J^*$ are isomorphisms and each is the inverse of the other.*

**Proof.** It follows immediately from Theorems 3.7 and 3.10.

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# IMPROVED MAC BASED DIFFERENTIAL FAULT ANALYSIS OF GRAIN-128a

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**ABSTRACT :** Differential Fault Attack (DFA) on stream ciphers is an active field of research. However, only two differential fault attacks were reported on Grain-128a so far. Moreover, among these two, only the scheme proposed in Banik et al. [3] targeted the cipher with MACs corresponding to the massages chosen by the adversary. But the attack strategy of [3] required huge number of fault injections and invocations of MAC generation routines. To be specific, it required less than $2^{11}$ fault injections and invocations of less than $2^{12}$ MAC generation routines. In this current paper we propose an efficient MAC based DFA on Grain-128a. To the best of our knowledge, the proposed paper, for the first time shows that under certain situations the MAC generation mechanism of Grain-128a reveals all suppressed pre-output bits. Once the suppressed pre-output bits are obtained by the adversary, SAT solvers are used to obtain the secret key. Our proposed attack is achieved just by observing the correct and faulty MACs of certain chosen messages (MAC for empty message is not required) with no more than 55 fault injections and no more than 60 re-keying (invoking MAC generation routine each time) of the cipher device. This is a significant improvement over [3]. Moreover, by allowing random unknown single bit faults at both the LFSR and the NFSR, we relax the fault model as considered in [3].

**Keywords :** Stream Cipher, Differential Fault Attack, MAC, Grain-128a, SAT Solver.

## 1. INTRODUCTION

Grain-128a [2] is the successor of Grain-128 in the Grain family of ciphers. It has low gate count, a low power consumption and a small chip area. Authors of Grain-128a also claimed that Grain-128a offers better scurity than any existing 128 bit cipher with the added possibility of authentication. Further after the publication of Grain-128a, the eSTREAM finalist Grain-128 was no longer recommended. Mode of operation (i.e., whether authentication is mandatory or not) of Grain-128a depends on the first bit of the IV. Up to now, only two DFAs ([3] and

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[11]) were proposed on Grain-128a. In the DFA model, faults are injected into the internal state of the cipher and from the difference of the normal and the faulty outputs, information about the internal state is partially or completely deduced [6], [4], [5], [10], [11], [8], [7], [9]. Faults can be injected in a register by under-powring and power spike, clock glitch, temperature attack, optical attack, electromagnetic (EM) fault injection, etc. Effect of the first three methods can hardly be focused to a particular part of the device. On the other hand, optical and EM methods can affect a very restricted area. Banik et al. [3] proposed a DFA on Grain-128a that recovers the secret key using MACs. It requires less than $2^{11}$ fault injections and invocations of less than $2^{12}$ MAC generation routines. Also it was assumed that the cipher device can be re-keyed with the original key-IV or the original key and different IVs. Sarkar et al. [11] proposed a DFA on Grain-128a in which normal (fault free) and faulty keystreams (due to single bit faults) were used to recover the key-IV (and requires 10 faults). It was assumed that the cipher device can be re-keyed with the same key-IV. This paper proposes a DFA on Grain-128a using MACs only and assuming that the cipher device can be re-keyed with the same key-IV. In this case our attack strategy recovers the secret key and IV with no more than 55 faults and no more than 60 MAC generation calls.

## 2. DESCRIPTION OF GRAIN-128a

The Grain-128a [1, 2] cipher consists of a 128-bit non-linear feedback shift register (NFSR) $X$ and a 128-bit linear feedback shift register (LFSR) $Y$. The NFSR and LFSR together represent the internal state of the cipher.

At round $i$ ($\geq -256$), the internal state $S_i$ of the Grain-128a cipher is given by,

$$S_i = \underbrace{(x_i, \ldots, x_{i+127}}_{X_i} \mid \underbrace{y_i, \ldots, y_{i+127})}_{Y_i},$$

where $X_i = (x_i, \ldots, x_{i+127})$ and $Y_i = (y_i, \ldots, y_{i+127})$ respectively denote the inner states of $X$ and $Y$.

**Key Loading algorithm (KLA).** The 128-bit secret *key* $(k_0, \ldots, k_{127})$ and 96-bit $IV$ $(IV_0, \ldots, IV_{95})$ are used to initialize the initial state (at round $i = -256$) as follows:

$$S_{-256} = (\underbrace{k_0,\ldots,k_{127}}_{X} \mid \underbrace{IV_0,\ldots,IV_{95},1,\ldots,1,0}_{Y}).$$

**Key scheduling Algorithm (KSA).** During rounds $i = -256$ to $-1$ (called the KSA rounds), the registers $X$ and $Y$ are respectively updated by $x_{i+128} = z_i + f(X_i, Y_i)$ and $y_{i+128} = z_i + g(Y_i)$ where,

$f(X_i, Y_i) = y_i + x_i + x_{i+26} + x_{i+56} + x_{i+91} + x_{i+96} + x_{i+3}x_{i+67} + x_{i+11}x_{i+13} + x_{i+17}x_{i+18} + x_{i+27}x_{i+59} + x_{i+40}x_{i+48} + x_{i+61}x_{i+65} + x_{i+68}x_{i+84} + x_{i+88}x_{i+92}x_{i+93}x_{i+95} + x_{i+22}x_{i+24}x_{i+25} + x_{i+70}x_{i+78}x_{i+82}$,

$g(Y_i) = y_i + y_{i+7} + y_{i+38} + y_{i+70} + y_{i+81} + y_{i+96}$

and $z_i = h(X_i, Y_i)$ is given by,

$h(X_i, Y_i) = x_{i+2} + x_{i+15} + x_{i+36} + x_{i+45} + x_{i+64} + x_{i+73} + x_{i+89} + y_{i+93} + x_{i+12}x_{i+95}y_{i+94} + x_{i+12}y_{i+8} + y_{i+13}y_{i+20} + x_{i+95}y_{i+42} + y_{i+60}y_{i+79}$.

Here $f(X_i, Y_i)$, $g(Y_i)$ and $h(X_i, Y_i)$ are respectively called the update function of NFSR, the update function of LFSR and the pre-output function. The bit $z_i$ is called the pre-output bit generated at the round $i$. During KSA rounds $i = -256$ to $-1$, the pre-output bit is fed back and XOR-ed with the input, both to the NFSR and to the LFSR. This pseudo-randomizes the internal state.

**Pseudo-Random keystream Generation Algorithm (PRGA).** After the KSA, in the PRGA rounds ($i \geq 0$), the pre-output bits are no longer XORed to the inputs of NFSR or LFSR and the registers $X$ and $Y$ are respectively updated by $x_{i+128} = f(X_i, Y_i)$ and $y_{i+128} = g(Y_i)$. The pre-output stream $z$ produced at the PRGA rounds is $z_0, z_1, z_2, \ldots$ and is used for keystream and MAC generation (optional).

**Modes of Operation.** Grain-128a supports two different modes of operation: with and without authentication. Authentication is mandatroy when $IV_0 = 1$, and forbidden when $IV_0 = 0$.

**Keystream Generation.** With $IV_0 = 0$, the output function is defined as simply $w_i = z_i$ ($i \geq 0$), meaning all the pre-output bits $z_0, z_1, z_2, \ldots$ are used directly as keystream. With $IV_0 = 1$, the output function is defined as $w_i = z_{64+2i}$ ($i \geq 0$), meaning that the cipher picks every second

bit as output of the cipher after skipping the first 64 bits. Those 64 initial bits and the other half will be used for authentication. In either case $w_0$, $w_1$, $w_2$, ... represents the generated keystream. When $IV_0 = 1$, the pre-output bits $z_0, ..., z_{63}$ and $z_{65}, z_{67}, z_{69}, ...$ are all suppressed. This pre-out stream $z_0, ..., z_{63}$ and then $z_{65}, z_{67}, z_{69}, ...$ will be denoted by $(z_0, ..., z_{63}) \| (z_{65}, z_{67}, z_{69}, ...)$ and will be called the *suppressed keystream*. We will denote the suppressed pre-output bits by $(s_0, s_1, s_2, ...)$ where $s_i = z_i$, for $0 \le i \le 63$ and $s_i = z_{65+2i}$ for $i \ge 0$.

Thus we are considering three bit-streams, namely

1. the pre-output stream $z = (z_0, z_1, z_2, ...)$,
2. the normal keystream $w = (w_0, w_1, w_2, ...)$ and
3. the suppressed keystream $s = (s_0, s_1, s_2, ...)$.

**MAC Generation Algorithm (MGA).** We assume a message of length $L$ defined by the bits $m_0, ..., m_{L-1}$. In this case $m_L = 1$ must be used as a padding. In order to provide authentication, two registers called the accumulator and the shift register of size 32 bits each, are used. The content of the accumulator at time $t$ is denoted by $a_t^0, ..., a_t^{31}$. The content of the shift register is denoted by $r_t, ..., r_{t+31}$. The accumulator is initialized through $a_0^j = z_j$, $0 \le j \le 31$, and the shift register is initialized through $r_t = z_{32+t}$, $0 \le t \le 31$. The shift register is updated as $r_{t+32} = z_{65+2t}$. The accumulator is updated as $a_{t+1}^j = a_t^j + m_t r_{t+j}$ for $0 \le j \le 31$ and $0 \le t \le L$. The final content of the accumulator, $(a_{L+1}^0, ..., a_{L+1}^{31})$, is used for authentication.

**Remark.** One should note that, the stream $(z_0, ...., z_{31}, r_0, r_1, r_2, ...)$ is identical with the suppressed keystream $(s_0, s_1, s_2, ...)$ and $a_{L+1}^j = z_j + \sum_{t=0}^{L} m_t r_{t+j}$ for $0 \le j \le 31$. We denote the message $m_0, ..., m_{L-1}$ simply by $msg = m_0 ... m_{L-1}$ and the generated MAC corresponding to the message simply by $\sigma(msg) = (a_{L+1}^0, ..., a_{L+1}^{31})$. The empty message will be denoted by $\phi$. Throughout the paper we shall now assume that $IV_0 = 1$ i.e., authentication is mandatory.

## 3. PROPOSED ATTACK ON GRAIN-128a : ATTACK MODEL, TOOLS AND DEFINITIONS

**Notations.** We shall use the following notations:

1. The empty set will be denoted by $\emptyset$.
2. For any two integers $a$ and $b$, we denote the set $\{x : x \text{ is an integer with } a \leq x \leq b\}$ simply by $[a, b]$.
3. The $i$-th element $u_i$ of any vector $u = (u_0, ..., u_{p-1})$ of length $p$ will be denoted by $u(i)$, $\forall i \in [0, p-1]$.
4. For any two vectors $u = (u_0, ..., u_{p-1})$ and $v = (v_0, ..., v_{p-1}) \in \{0, 1\}^p$ of equal length, we denote $u + v = (u_0 + v_0, ..., u_{p-1} + v_{p-1})$.
5. For any vector $u = (u_0, .., u_{p-1}) \in \{0, 1\}^p$ and any bit $b \in \{0, 1\}$ we denote $bu = (bu_0, ..., bu_{p-1})$. Also We denote $1u$ simply by $u$.
6. For any two vectors $u = (u_0, ..., u_{p-1})$ and $v = (v_0, ..., v_{p-1}) \in \{0, 1\}^p$, we denote $u \| v = (u_0, ..., u_{p-1}, v_0, ..., v_{p-1})$. Thus $\|$ represents the concatenation operator.

**Fault Location.** A single bit fault flips a register bit value. We assume that all faults are injected at the beginning of the PRGA round 0. If a fault flips a register bit value then its position will be called as the fault location.

**Attack model.** In this paper we assume that the adversary is given access to an Oracle which possesses the unknown key and IV. The adversary queries the Oracle for the following information:

1. MACs $\sigma(msg_0)$, ..., $\sigma(msg_{p-1})$ corresponding to the $p$ messages $msg_0$, ..., $msg_{p-1}$ chosen by the adversary, all generated by the normal (fault free) pre-output stream $z$.
2. MACs $\sigma^{\phi}(msg_0)$, ..., $\sigma^{\phi}(msg_{p-1})$ corresponding to the same $p$ messages $msg_0$, ..., $msg_{p-1}$, all generated by the faulty pre-output stream $z^{\phi}$, for $m$ (chosen by the adversary) fault locations $\phi \in \{\phi_0, ..., \phi_{m-1}\}$, where the $m$ distinct fault locations $\phi_0, ..., \phi_{m-1}$ are chosen by the Oracle and are not given to the adversary.

In this case, $m$ (chosen by the adversary) register locations are needed to disturb. A total $mp$ number of faults are required to be injected and total $(m + 1)p$ re-keying are needed. The adversary only has $\sigma(msg_0)$, ..., $\sigma(msg_{p-1})$ and $\sigma^{\phi}(msg_0)$, ..., $\sigma^{\phi}(msg_{p-1})$, $\forall \phi \in \{\phi_0, ..., \phi_{m-1}\}$ for the messages $msg_0$, ..., $msg_{p-1}$ chosen by the adversary. Note that $\phi_0$, ..., $\phi_{m-1}$ are not known to the adversary. In this paper we want to minimize $mp$.

**Attack strategy in a nutshell:** With the information obtained from the Oracle, the adversary first finds the suppressed keystream bits by solving some algebraic equations, then identifies the fault locations $\phi_0, ..., \phi_{m-1}$. After that the adversary passes the available information to the SAT solver in SAGE and extracts the fault free internal state at round $i = 0$. Now by iterating backwards the key-IV can be recovered.

## 4. SUPPRESSED KEYSTREAM DETERMINATION

We now agree to denote the $k$-bit message $\underbrace{0...0}_{k}$ simply by $0_k$.

Let us consider the 4 messages 1, 0, $0_{32}$ and $0_{64}$. Note that the corresponding MACs are available to the adversary.

One should note that,

$$\sigma(1) = (z_0 + r_0 + r_1, z_1 + r_1 + r_2, ..., z_{30} + r_{30} + r_{31}, z_{31} + r_{31} + r_{32}),$$

$$\sigma(0) = (z_0 + r_1, z_1 + r_2, ..., z_{30} + r_{31}, z_{31} + r_{32}),$$

$$\sigma(0_{32}) = (z_0 + r_{32}, ..., z_{31} + r_{63}),$$

$$\sigma(0_{64}) = (z_0 + r_{64}, ..., z_{31} + r_{95}).$$

Thus $\sigma(1)$ and $\sigma(0)$ give $r_0, ..., r_{31}$ and from $\sigma(0)$ we obtain $z_0, ..., z_{30}$.

Again from $\sigma(0_{32})$ and $\sigma(0)$ we obtain $z_{31}$ and $r_{32}, ..., r_{63}$.

Similarly $\sigma(0_{64})$ and $\sigma(0_{96})$ give $r_{64}, ..., r_{127}$.

Thus we have $z_0, ..., z_{31}, r_0, ..., r_{127}$ i.e., we have the suppressed keystream bits $(z_0, ..., z_{63})||(z_{65}, z_{67}, z_{69}, ..., z_{255})$. And these bits are obtained without using the empty message $\phi$.

The suppressed keystream bits for higher PRGA rounds (in a chunk of 32) could be obtained by observing the MACs of $0_{32L}$ for $L > 2$ iteratively.

## 5. FAULT SIGNATURE

Let for some fixed key-IV, $(s_0, ..., s_{n-1})$ be the suppressed keystream of length $n$. Let us consider a fault location $\phi$. Let $(s_0^{\phi}, ..., s_{n-1}^{\phi})$ be the faulty suppressed keystream of length $n$ under the fault at location $\phi$.

In this case the bitwise XOR difference of the normal and faulty suppressed keystreams will be called the XOR differential suppressed keystream of length $n$ and will be denoted by $d^{\psi,n} = (d_0^{\psi}, ..., d_{n-1}^{\psi})$.

We now define the following sets,

$$sig_{\phi}^{1} = \{i \in [0, n-1] : \Pr[d_i^{\psi} = 1] = 1\};$$

$$sig_{\phi}^{0} = \{i \in [0, n-1] : \Pr[d_i^{\psi} = 0] = 1\};$$

$$sig_{\phi}^{=} = \{\{i, j\} : i, j \in [0, n-1] \text{ and } \Pr[d_i^{\psi} + d_j^{\psi} = 0] = 1\}$$

$$sig_{\phi}^{\neq} = \{\{i, j\} : i, j \in [0, n-1] \text{ and } \Pr[d_i^{\psi} + d_j^{\psi} = 1] = 1\}$$

In this case $sig_{\phi} = (sig_{\phi}^{1}, sig_{\phi}^{0}, sig_{\phi}^{=}, sig_{\phi}^{\neq})$ will be called the signature of the fault location $\phi$ and mathematically represents the occurrence of some certain events under the fault $\phi$. One could use large number of experiments in order to compute the fault signatures.

## 6. FAULT LOCATION IDENTIFICATION

Let at the online stage, a single bit fault is injected at an unknown location $\psi$. The adversary will use the following instruction for detecting fault location:

Obtain the XOR differential suppressed keystream $d^{\psi,n} = (d_0^{\psi}, ..., d_{n-1}^{\psi})$.

Compute, $support^1 = \{i \in [0, n-1] : d_i^{\psi} = 1\}$.

Compute, $support^0 = \{i\ [0, n-1] : d_i^{\psi} = 0\}$.

Compute, $pf^1 = \{\phi : \phi \in [0, 255]$ and $sig_\phi^1 \subseteq support^1\}$.

Compute, $pf^0 = \{\phi \in pf^1 : sig_\phi^0 \subseteq \text{support}^0\}$.

Compute, $pf^= = \{\phi \in pf^0 : sig_\phi^= \neq \emptyset$ and $\{i, j\} \in sig_\phi^= \Rightarrow d_i^{\psi} = d_j^{\psi}\} \cup \{\phi \in pf^0 : sig_\phi^= = \emptyset\}$.

Compute, $pf^{\neq} = \{\phi \in pf^= : sig_\phi^{\neq} \neq \emptyset$ and $\{i, j\} \in sig_\phi^{\neq} \Rightarrow d_i^{\psi} + d_j^{\psi} = 1\} \cup \{\phi \in pf^= : sig_\phi^{\neq} \neq \emptyset\}$.

Define, $pf = pf^{\neq}$.

The basic idea is to check whether the pre-computed pattern (signature) of a fault location occurs in the XOR differential suppressed keystream. If the pattern due to a fault location occurs in the XOR differential suppressed keystream, then it is a possible fault location, otherwise we reject it. It should be noted that from the construction it immediately follows that the actual fault location $\psi \in pf$. Now if $pf$ is singleton then, $\psi$ is uniquely determined.

**Experimental Result.** In $2^{20}$ trials, we generated key-IV randomly and simulated the injection of random single bit faults to the internal state at PRGA round 0. Experimental results show that, by considering the first 256 PRGA rounds, the fault location could be identified uniquely with a probability of 0.8.

## 7. STATE RECOVERY USING SAT SOLVERS

The adversary wishes to recover the internal state of the cipher at the PGRA round 0 and starts with the following information:

1. $m$ fault locations, given by $\beta = (\phi_0, \phi_1, ..., \phi_{m-1})$.
2. normal (fault free) suppressed keystream $s = (s_0, ..., s_{n-1})$ of length $n$.
3. $m$ faulty keystreams $s^0, ..., s^{m-1}$ each of length $n$, where $s^l$ is the faulty suppressed keystream due the fault $\phi_l$.

Let the fault free internal state at the PRGA round $i$ ($\geq 0$) be $S_i = (X_i, Y_i)$ where $X_i = (x_i, \ldots, x_{i+127})$ and $Y_i = (y_i, \ldots, y_{i+127})$, the internal state at PRGA round 0 being $S_0 = (x_0, \ldots, x_{127}, y_0, \ldots, y_{127})$. We treat each $x_i$ and $y_i$ as variables and consider the PRGA rounds 0, ..., $k$. Corresponding to each pre-output bit $z_i$, we introduce two new variables $x_{i+128}$ ($i \geq 0$) and obtain the following two equations: $x_{i+128} = f(X_i, Y_i), y_{i+128} = g(Y_i)$. We also consider the equation $z_i = h(X_i, Y_i)$ when $z_i$ is a suppressed keystream bit.

Let us now consider the fault location $\phi_l$. Since the cipher device is re-keyed before each fault injection, after the fault injection, if the faulty internal state at PRGA round $i$ be $S_i^j$ then at the targeted fault injection PRGA round 0 we have, $S_0^j(e) = S_0(e) + 1$, $\forall e \in \phi_l$ and $S_0^j(e) = S_0(e)$, $\forall e \in [0, 255] \setminus \{\phi_l\}$. Again corresponding to each pre-output bit $z_i$, we introduce two new variables $x_{i+128}^j$, $y_{i+128}^j$ ($i \geq 0$) and obtain three more equations.

Now the system of polynomial equations are simply passed on to the SAT solver in SAGE for extracting solution for the variables $x_0, \ldots, x_{127}, y_0, \ldots, y_{127}$. Now by reversing backwards the secret key could be recovered.

Experimental results (a total of 100 experiments) show that, for $m = 11$, $k = 255$ (MACs corresponding to the messages 1, 0, $0_{32}$, $0_{64}$ and $0_{96}$) the state could be recovered with probability 1.0 in an average time of 84.36 seconds.

## 8. CONCLUSION

In this current paper we propose an efficient MAC based DFA on Grain-128a. To the best of our knowledge, the proposed paper, for the first time, shows that under certain situations the MAC generation mechanism of Grain-128a reveals all suppressed pre-output bits. Once the suppressed pre-output bits are obtained by the adversary, SAT solvers are used to obtain the secret key. Our proposed attack is achieved just by observing the correct and faulty MACs of certain chosen messages (MAC for empty message is not requred) with no more than 55 fault injections and no more than 60 re-keying (invoking MAC generation routine each time) of the cipher device. This is a significant improvement over [3].

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# ON SEPARATION AXIOMS WEAKER AND STRONGER THAN REGULARITY AND NORMALITY VIA GRILLS

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**ABSTRACT :** In this paper, a few types of separation axioms for topological spaces are introduced and studied in terms of grills; of these, one class is contained in and another contains the class of regular spaces. Two other types, one being weaker and another stronger than normality, are also defined and investigated along similar line.

**Key words :** Grill, topology $\tau_{\mathcal{G}}$, $\mathcal{G}$-$g$-closed sets, $\mathcal{G}$-$g$-regular, $\mathcal{G}$-$g$-normal.

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## 1. INTRODUCTION AND PRELIMINARIES

Different neighbouring forms of standard separation properties like regularity and normality, are being studied with interest for a long time. Munshi in [7] studied two kinds of separation axioms, called $g$-regularity and $g$-normality, stronger than regularity and normality respectively. Our intention in this paper is to follow the idea of Munshi towards introduction of some other separation properties by use of the concept of grills.

In 1947, the idea of grill was first introduced by Choquet [2], a detailed study of which was sbsequently undertaken by Thron [13] and many others. The definition of grill on a topological space $X$ as given by Choquet [2], goes as follows:

**Definition 1.1.** [2] A nonempty collection $\mathcal{G}$ of nonempty subsets of a topological space $X$ is called a grill if

(i) $A \in \mathcal{G}$ and $A \subseteq B \subseteq X \Rightarrow B \in \mathcal{G}$, and (ii) $A, B \subseteq X$ and $A \cup B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.

For a grill $\mathcal{G}$ on a topological space $(X, \tau)$, Roy and Mukherjee [11] defined an operator $\Phi$ from the power set $P(X)$ to $P(X)$ in the following way : For any $A \subseteq X$, , $\Phi(A) = \{x \in X : U \cap A \in \mathcal{G}$ for every open set $U$ containing $x\}$. It was also shown in [11] that

the map $\Psi : P(X) \to P(X)$, given by $\Psi(A) = A \cup \Phi(A)$ (for $A \subseteq X$), is a Kuratowski closure operator giving rise to a topology $\tau_{\mathcal{G}}$ (say) on $X$, finer than $\tau$. Thus a subset $A$ of $X$ is $\tau_{\mathcal{G}}$-closed if $\Psi(A) = A$ or equivalently if $\Phi(A) \subseteq A$. A topological space endowed with a grill $\mathcal{G}$ on $X$, denote by $(X, \tau, \mathcal{G})$ will be called a grill topological space.

In this paper we introduce and study certain types of separation axioms, termed $\mathcal{G}$-$g$-regular, $g$-$\mathcal{G}$-regular, $\mathcal{G}$-$g$-normal and $g$-$\mathcal{G}$-normal spaces by using $\mathcal{G}$-$g$-open sets introduced in [5] and obtain some characterizations of these spaces. Also we obtain some preservation theorems for $\mathcal{G}$-$g$-regular and $\mathcal{G}$-$g$-normal spaces in Section 4.

In what follows, by a space $X$ we shall mean a topological space $(X, \tau)$. For any $A \subseteq X$, int$(A)$ and cl$(A)$ will respectively stand for the interior and closure of $A$ in $(X, \tau)$. Again, $\tau_{\mathcal{G}}$-cl$(A)$ and $\tau_{\mathcal{G}}$-int$(A)$ will respectively mean the closure and interior of $A$ in $(X, \tau_{\mathcal{G}})$. Similarly, whenever we say that a subset $A$ of a space $X$ is open (or closed) in $X$, these are meant to be so in $(X, \tau)$. For open and closed sets with respect to any other topology on $X$ e.g. $\tau_{\mathcal{G}}$, we shall write '$\tau_{\mathcal{G}}$-open' and '$\tau_{\mathcal{G}}$-closed'. A subset $A$ of a space $(X, \tau)$ is said to be preopen [6] ($\alpha$-open [8]) if $A \subseteq \text{int}(\text{cl}(A))$ (resp. $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$. The family of all $\alpha$-open sets in $(X, \tau)$, denoted by $\tau^\alpha$, is known to be a topology on $X$ finer than $\tau$, and the closure of $A$ in $(X, \tau^\alpha)$ is denoted by $\alpha$-cl$(A)$. We now recall a few definitions and results as prerequisites.

**Definition 1.2.** A subset $A$ of a space $(X, \tau)$ is said to be $g$-closed [3] ($\alpha g$-closed [4]) if cl$(A) \subseteq U$ (resp. $\alpha$-cl$(A) \subseteq U$) whenever $A \subseteq U$ and $U$ is open. The complement of a $g$-closed ($\alpha g$-closed) set is called a $g$-open (resp. an $\alpha g$-open) set.

**Definition 1.3.**[7] A space $(X, \tau)$ is said to be $g$-regular if for each $x \in X$ and each $g$-closed set $F$ with $x \notin F$, there exist disjoint open sets $U$ and $V$ such that $x \in U$ and $F \subseteq V$.

**Definition 1.4.**[7] A space $(X, \tau)$ is said to be $g$-normal if for each pair of disjoint $g$-closed sets $F$ and $K$, there exist disjoint open sets $U$ and $V$ such that $F \subseteq U$ and $K \subseteq V$.

Let us now define a space $(X, \tau)$ to be $\alpha g$-normal if for each pair of disjoint $\alpha g$-closed sets $F$ and $K$, there exist disjoint open sets $U$ and $V$ such that $F \subseteq U$ and $K \subseteq V$.

**Theorem 1.5.**[5] Let $\mathcal{G}$ be a grill on a space $(X, \tau)$ such that $PO(X)\setminus\{\phi\} \subseteq \mathcal{G}$. Then $\tau_{\mathcal{G}} \subseteq \tau^\alpha$, where $PO(X)$ denotes the collection of all preopen sets in $(X, \tau)$.

**Definition 1.6.**[5] Let $(X, \tau)$ be a topological space and $\mathcal{G}$ be a grill on $X$. Then a subset $A$ of $X$ is said to be $g$-closed with respect to the grill $\mathcal{G}$ ($\mathcal{G}$-$g$-closed, for short) if $\Phi(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

A subset $A$ of $X$ is said to be $\mathcal{G}$-$g$-open if $X \setminus A$ is $\mathcal{G}$-$g$-closed.

**Theorem 1.7.**[5] Let $(X, \tau)$ be a topological space and $\mathcal{G}$ be a grill on $X$. Then for a subset $A$ of $X$, the following are equivalent:

(a) $A$ is $\mathcal{G}$-$g$-closed.

(b) $\tau_{\mathcal{G}}$-cl$(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open.

(c) For all $x \in \tau_{\mathcal{G}}$-cl$(A)$, cl$(\{x\}) \cap A \neq \phi$.

(d) $\tau_{\mathcal{G}}$-cl$(A) \setminus A$ contains no nonempty closed set of $(X, \tau)$.

(e) $\Phi(A) \setminus A$ contains no nonempty closed set of $(X, \tau)$.

Corresponding to any nonempty subset $A$ of $X$, a typical grill $[A]$ on $X$ was defined in [12] in the following manner.

**Definition 1.8.** Let $X$ be a space and $(\phi \neq)\ A \subseteq X$. Then

$$[A] = \{B \subseteq X : A \cap B \neq \phi\}$$

is a grill on $X$, called the principal grill generated by $A$.

**Remark 1.9.** It is shown in [5] that a $g$-closed set is $\mathcal{G}$-$g$-closed but not conversely. However, in the case of principal grill $[X]$ generated by $X$, it is known [12] that $\tau = \tau_{[X]}$, so that any $[X]$-$g$-closed set becomes simply a $g$-closed set and vice-versa.

**Theorem 1.10.**[5] Let $(X, \tau)$ be a topological space and $A \subseteq X$. Then $\tau_{\mathcal{G}_\delta} = \tau^\alpha$ and hence a subset $A$ of $X$ is $\mathcal{G}_\delta$-$g$-closed iff $A$ is $\alpha g$-closed where $\mathcal{G}_\delta$ is the grill on $X$ given by $\mathcal{G}_\delta = \{A \subseteq X : \text{int}(\text{cl}(A)) \neq \phi\}$.

**Theorem 1.11.** [5] Let $\mathcal{G}$ be a grill on a space $(X, \tau)$. Then $A\ (\subseteq X)$ is $\mathcal{G}$-$g$-open iff $F \subseteq \tau_{\mathcal{G}}$-int$(A)$ whenever $F \subseteq A$ and $F$ is closed.

**Theorem 1.12.**[5] For any grill $\mathcal{G}$ on a space $(X, \tau)$ the following are equivalent:

(a) Every subset of $X$ is $\mathcal{G}$-$g$-closed.

(b) Every open subset of $(X, \tau)$ is $\tau_{\mathcal{G}}$-closed.

## 2. $\mathcal{G}$-$g$-REGULAR AND $\mathcal{G}$-G-REGULAR SPACES

**Definition 2.1.** Let $(X, \tau)$ be a topological space and $\mathcal{G}$ be a grill on $X$. Then $(X, \tau)$ is said to be $\mathcal{G}$-$g$-regular if for each $x \in X$ and each $\mathcal{G}$-$g$-closed set $F$ with $x \notin F$, there exist disjoint open sets $U$ and $V$ such that $x \in U$ and $F \subseteq V$.

**Remarks 2.2.** Since every closed set is $\mathcal{G}$-$g$-closed for any grill $\mathcal{G}$ on $X$, every $\mathcal{G}$-$g$-regular space is regular. But the converse is false as is shown by the following example.

**Example 2.3.** Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b, c\}, X\}$. Then $(X, \tau)$ is regular space but it is not $\mathcal{G}$-$g$-regular for any grill $\mathcal{G}$ on $X$. In fact, for any grill $\mathcal{G}$ on $X$, $F = \{b\}$ is $\mathcal{G}$-$g$-closed and $c \notin F$, but there are no disjoint open sets which contain $c$ and $F$.

**Remark 2.4.** In the case of principal grill $[X]$ generated by $X$, it is obvious that any $[X]$-$g$-regular space becomes simply a $g$-regular space and conversely (refer to Remark 1.9).

**Theorem 2.5.** Let $\mathcal{G}$ be a grill on a space $(X, \tau)$. Then the following are equivalent :

(a) $(X, \tau)$ is $\mathcal{G}$-$g$-regular.

(b) For each $x \in X$ and each $\mathcal{G}$-$g$-open set $U$ containing $x$, there exists an open set $V$ in $X$ such that $x \in V \subseteq \mathrm{cl}(V) \subseteq U$.

(c) For each $x \in X$ and each $\mathcal{G}$-$g$-closed set with $x \notin F$, there exist disjoint open sets $U$ and $V$ such that $x \in U$ and $\tau_{\mathcal{G}}$-$\mathrm{cl}(F) \subseteq V$.

(d) For each $\mathcal{G}$-$g$-closed set $F$ and each point $x \in X \setminus F$, there exist open sets $U$ and $V$ of $X$ such that $x \in U$, $F \subseteq V$ and $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \phi$.

**Proof.** (a) $\Rightarrow$ (*b*) : For a given $x \in X$, let $U$ be any $\mathcal{G}$-$g$-open set containing $x$. Then by hypothesis, there exist disjoint open sets $V$ and $W$ such that $x \in V$ and $X \setminus U \subseteq W$. Now $V \cap W = \phi$ $\Rightarrow \mathrm{cl}(V) \subseteq X \setminus W \subseteq U$. Thus $x \in V \subseteq cl(V) \subseteq U$.

(*b*) $\Rightarrow$ (*a*) : Let $x \in X$ and $F$ be a $\mathcal{G}$-$g$-closed set with $x \notin F$. Then $x \in X \setminus F$ and so by

($a$), there exists an open set $V$ such that $x \in V \subseteq \text{cl}(V) \subseteq X \setminus F$. Put $W = X \setminus \text{cl}(V)$. Then $V$ and $W$ are disjoint open sets such that $x \in V$, $F \subseteq W$. Hence $(X, \tau)$ is a $\mathcal{G}$-$g$-regular space.

($a$) $\Rightarrow$ ($c$) : Let $x \in X$ and $F$ be a $\mathcal{G}$-$g$-closed set not containing $x$. Then by ($a$), there exist disjoint open sets $U$ and $V$ in $X$ such that $x \in U$ and $F \subseteq V$. Since $F$ is $\mathcal{G}$-$g$-closed, $\Phi(F) \subseteq V$ i.e., $\tau_{\mathcal{G}}\text{-cl}(F) \subseteq V$.

($c$) $\Rightarrow$ ($d$) : Let $x \in X$ and $F$ be a $\mathcal{G}$-$g$-closed set not containing $x$. Then by ($c$), there exist disjoint open sets $W$ and $V$ such that $x \in W$ and $\tau_{\mathcal{G}}\text{-cl}(F) \subseteq V$. Also we have $\text{cl}(V) \cap W = \phi$. Now $\text{cl}(V)$ is $\mathcal{G}$-$g$-closed and $x \notin \text{cl}(V)$. Then again by ($c$), there exist open sets $G$ and $H$ in $X$ such that $x \in G$, $\tau_{\mathcal{G}}\text{-cl}(\text{cl}(V)) \subseteq H$ and $G \cap H = \phi$ i.e., $x \in G$, $\text{cl}(V) \subseteq H$ and $G \cap H = \phi$ and hence $\text{cl}(G) \cap H = \phi$. Now put $U = W \cap G$, then $U$ and $V$ are open subsets of $X$ such that $x \in U$, $F \subseteq V$ and $\text{cl}(U) \cap \text{cl}(V) = \phi$.

($d$) $\Rightarrow$ ($a$) : It is clear.

If we take $\mathcal{G} = [X]$ in the above theorem, then by using Remarks 1.9 and 2.4, we have the following result of Noiri and Popa [10].

**Corollary 2.6.** For the topological space $(X, \tau)$, the following are equivalent :

($a$) $(X, \tau)$ is $g$-regular.

($b$) For each $x \in X$ and each $g$-open set $U$ containing $x$, there exists an open set $V$ in $X$ such that $x \in V \subseteq \text{cl}(V) \subseteq U$.

($c$) For each $x \in X$ and each $g$-closed set with $x \notin F$, there exist disjoint open sets $U$ and $V$ such that $x \in U$ and $\text{cl}(F) \subseteq V$.

($d$) For each $g$-closed set $F$ and each point $x \in X \setminus F$, there exist open sets $U$ and $V$ of $X$ such that $x \in U$, $F \subseteq V$ and $\text{cl}(U) \cap \text{cl}(V) = \phi$.

We now define another type of separation axiom in grill topological spaces as follows:

**Definition 2.7.** Let $\mathcal{G}$ be a grill on a space $(X, \tau)$. Then $(X, \tau)$ is said to be $g$-$\mathcal{G}$-regular if for each point $x \in X$ and for each closed set $F$ with $x \notin F$ there exist disjoint $\mathcal{G}$-$g$-open sets $U$ and $V$ such that $x \in U$ and $F \subseteq V$.

**Remark 2.8.** It is easy to see that the following implication diagram holds :

$$\mathcal{G}\text{-}g\text{-regularity} \Rightarrow g\text{-regularity} \Rightarrow \text{regularity} \Rightarrow g\text{-}\mathcal{G}\text{-regularity}.$$

We show below that a $g$-$\mathcal{G}$-regular space need not be regular, and hence neither $g$-regular nor $\mathcal{G}$-$g$-regular.

**Example 2.9.** Consider a grill $\mathcal{G} = \{\{b\}, \{a, b\}, \{b, c\}, X\}$ on a space $(X, \tau)$ where $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$. It is easy to see that $(X, \tau)$ is not regular but it is $g$-$\mathcal{G}$-regular. In fact, $\Phi(\{a\}) = \phi$, $\Phi(\{a, b\}) = \{b\}$, $\Phi(\{a, c\}) = \phi$. Thus every open set is $\tau_{\mathcal{G}}$-closed and so by Theorem 1.12, every subset of $X$ is $\mathcal{G}$-$g$-closed and hence every subset of $X$ is $\mathcal{G}$-$g$-open. Hence $(X, \tau)$ is $g$-$\mathcal{G}$-regular.

**Theorem 2.10.** Let $\mathcal{G}$ be a grill on a space $(X, \tau)$. Then the following are equivalent : (*a*) $X$ is $g$-$\mathcal{G}$-regular.

(*b*) For each open set $V$ containing $x \in X$, there exists an open set $U$ such that $x \in U \subseteq \tau_{\mathcal{G}}\text{-cl}(U) \subseteq V$.

**Proof.** (*a*) $\Rightarrow$ (*b*) : Let $V$ be any open set in $(X, \tau)$ containing a point $x$ of $X$. Then by hypothesis, there exist disjoint $\mathcal{G}$-$g$-open sets $U$ and $W$ such that $x \in U$ and $X \setminus V \subseteq W$. Since $W$ is $\mathcal{G}$-$g$ open and $X \setminus V \subseteq W$ with $X \setminus V$ closed, we have by Theorem 1.11, $X \setminus V \subseteq \tau_{\mathcal{G}}\text{-int}(W)$. Now $U \cap W = \phi \Rightarrow U \cap \tau_{\mathcal{G}}\text{-int}(W) = \phi \Rightarrow \tau_{\mathcal{G}}\text{-cl}(U) \subseteq X \setminus \tau_{\mathcal{G}}\text{-int}(W) \subseteq V$. Thus $x \in U \subseteq \tau_{\mathcal{G}}\text{-cl}(U) \subseteq V$.

(*b*) $\Rightarrow$ (*a*) : Let $F$ be a closed set in $X$ not containing $x \in X$. Then by hypothesis, there exists a $\mathcal{G}$-$g$-open set $U$ such that $x \in U \subseteq \tau_{\mathcal{G}}\text{-cl}(U) \subseteq X \setminus F$. Put $V = X \setminus \tau_{\mathcal{G}}\text{-cl}(U)$. Then $U$ and $V$ are disjoint $\mathcal{G}$-$g$-open sets such that $x \in U$ and $F \subseteq V$. Hence $(X, \tau)$ is a $g$-$\mathcal{G}$-regular space.

Let us now recall the following result from [5].

**Theorem 2.11.** Let $\mathcal{G}$ be a grill on a $T_1$-space $(X, \tau)$ such that $PO(X) \setminus \{\phi\} \subseteq \mathcal{G}$. Then the following are equivalent :

(a) $X$ is regular.

(b) For each closed set $F$ and each $x \in X \setminus F$, there exist disjoint $\mathcal{G}$-$g$-open sets $U$ and $V$ such that $x \in U$ and $F \subseteq V$.

(c) For each open set $V$ in $(X, \tau)$ and each point $x \in V$, there exists a $\mathcal{G}$-$g$-open set $U$ such that $x \in U \subseteq \tau_{\mathcal{G}}\text{-cl}(U) \subseteq V$.

Combining Theorems 2.10 and 2.11, we get the following result:

**Corollary 2.12.** Let $\mathcal{G}$ be a grill on a $T_1$-space $(X, \tau)$ such that $PO(X) \setminus \{\phi\} \subseteq \mathcal{G}$. Then $(X, \tau)$ is $g$-$\mathcal{G}$-regular iff it is regular.

## 3. $\mathcal{G}$-$g$-NORMAL AND $g$-$\mathcal{G}$-NORMAL SPACES

As in the last section, we introduce here two variant forms of normality, one being stronger and another weaker than normality.

**Definition 3.1.** Let $\mathcal{G}$ be a grill on a space $(X, \tau)$. Then $(X, \tau)$ is said to be $\mathcal{G}$-$g$-normal if for each pair of disjoint $\mathcal{G}$-$g$-closed sets $F$ and $K$, there exist disjoint open sets $U$ and $V$ in $X$ such that $F \subseteq U$ and $K \subseteq V$.

**Remark 3.2.** Since every closed set is $\mathcal{G}$-$g$-closed for any grill $\mathcal{G}$ on $X$, every $\mathcal{G}$-$g$-normal space in normal. But the converse is false as is shown by the following example.

**Example 3.3.** Consider a grill $\mathcal{G} = \{\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$ on a topological space $(X, \tau)$ where $X = \{a, b, c, d\}$, $\tau = \{\phi, \{b\}, X\}$. Then $(X, \tau)$ is normal but is not $\mathcal{G}$-$g$-normal. In fact, every open subset of $X$ is $\tau_{\mathcal{G}}$-closed and hence by Theorem 1.12 every subset of $X$ is $\mathcal{G}$-$g$-closed. Now $F = \{a, d\}$, $K = \{b, c\}$ are disjoint $\mathcal{G}$-$g$-closed sets, but they cannot be separated by disjoint open sets in $X$.

**Theorem 3.4.** Let $\mathcal{G}$ be a grill on a space $(X, \tau)$. Then $(X, \tau)$ is $\mathcal{G}$-$g$-normal iff for each $\mathcal{G}$-$g$-closed set $F$ and each $\mathcal{G}$-$g$-open set $U$ containing $F$, there exists an open set $V$ in $X$ such that $F \subseteq V \subseteq \text{cl}(V) \subseteq U$.

**Proof.** The straightforward proof is omitted.

If the principal grill $[X]$ takes the role of $\mathcal{G}$ in the above theorem, then we obtain the following characterizations of a $g$-normal space.

**Corollary 3.5.** A topological space $(X, \tau)$ is $g$-normal iff for each $g$-closed set $F$ and for any $g$-open set $U$ containing $F$, there exists an open set $V$ of $X$ such that $F \subseteq V \subseteq \mathrm{cl}(V) \subseteq U$.

**Theorem 3.6.** The following are equivalent for a grill topological space $(X, \tau, \mathcal{G})$:

(*a*) $X$ is $\mathcal{G}$-$g$-normal.

(*b*) For each pair of disjoint $\mathcal{G}$-$g$-closed sets $F$ and $K$, there exists an open set $U$ in $X$ containing $F$ such that $\mathrm{cl}(U) \cap K = \phi$.

(*c*) For each pair of disjoint $\mathcal{G}$-$g$-closed sets $F$ and $K$, there exist open sets $U$ and $V$ in $X$ such that $F \subseteq U$, $K \subseteq V$ and $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \phi$.

**Proof.** (*a*) $\Rightarrow$ (*b*) : Let $F$ and $K$ be two disjoint $\mathcal{G}$-$g$-closed sets. Then by Theorem 3.4, there exists an open set $U$ such that $F \subseteq U \subseteq \mathrm{cl}(U) \subseteq X \setminus K$. Thus for the open set $U$ we have, $F \subseteq U$ and $\mathrm{cl}(U) \cap K = \phi$.

(*b*) $\Rightarrow$ (*c*) : Let $F$ and $K$ be two disjoint $\mathcal{G}$-$g$-closed sets. Then by (*b*), there exists an open set $U$ in $X$ such that $F \subseteq U$ and $\mathrm{cl}(U) \cap K = \phi$. Again, since $K$ and $\mathrm{cl}(U)$ are disjoint $\mathcal{G}$-$g$-closed sets, by hypothesis there exists an open set $V$ such that $K \subseteq V$ and $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \phi$.

(*c*) $\Rightarrow$ (*a*) : Obvious.

If in the above theorem, we take $\mathcal{G} = [X]$, then by using Remark 1.9, we arrive at the following known result (viz Theorem 4.1 of Noiri and Popa [10]).

**Corollary 3.7.** For a topological space $X$, the following are equivalent :

(*a*) $X$ is $g$-normal.

(*b*) For each pair of disjoint $g$-closed sets $F$ and $K$, there exists an open set $U$ in $X$ containing $F$ such that $\mathrm{cl}(U) \cap K = \phi$.

(*c*) For each pair of disjoint $g$-closed sets $F$ and $K$, there exist open sets $U$ and $V$ in $X$ such that $F \subseteq U$, $K \subseteq V$ and $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \phi$.

If we set $\mathcal{G} = \mathcal{G}_\delta$ in Theorem 3.6, we get the following result.

**Corollary 3.8.** For a topological space $X$, the following are equivalent :

(*a*) $X$ is $\alpha g$-normal.

(*b*) For each pair of disjoint $\alpha g$-closed sets $F$ and $K$, there exists an open set $U$ in $X$ containing $F$ such that $\mathrm{cl}(U) \cap K = \phi$.

(*c*) For each pair of disjoint $\alpha g$-closed sets $F$ and $K$, there exist open sets $U$ and $V$ in $X$ such that $F \subseteq U$, $K \subseteq V$ and $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \phi$.

**Theorem 3.9.** Let $\mathcal{G}$ be a grill on a space $(X, \tau)$. If $F$ and $K$ are disjoint $\mathcal{G}$-$g$-closed sets of a $\mathcal{G}$-$g$-normal space $(X, \tau)$, then there exist disjoint open sets $U$ and $V$ in $X$ such that $\tau_{\mathcal{G}}\text{-cl}(F) \subseteq U$, $\tau_{\mathcal{G}}\text{-cl}(K) \subseteq V$.

**Proof.** Let $F$ and $K$ be two disjoint $\mathcal{G}$-$g$-closed sets of a $\mathcal{G}$-$g$-normal space $(X, \tau)$. Then there exist disjoint open sets $U$ and $V$ such that $F \subseteq U$ and $K \subseteq V$. Since $F$ is $\mathcal{G}$-$g$-closed and $F \subseteq U$, $\tau_{\mathcal{G}}\text{-cl}(F) \subseteq U$. Similarly, $\tau_{\mathcal{G}}\text{-cl}(K) \subseteq V$.

**Theorem 3.10.** Let $\mathcal{G}$ be a grill on a $\mathcal{G}$-$g$-normal space $(X, \tau)$. If $F$ is a $\mathcal{G}$-$g$-closed set and $V$ be a $\mathcal{G}$-$g$-open set in $X$ such that $F \subseteq V$, then there exists an open set $U$ in $X$ such that $F \subseteq \tau_{\mathcal{G}}\text{-cl}(F) \subseteq U \subseteq \tau_{\mathcal{G}}\text{-int}(V) \subseteq V$.

**Proof.** Since $F$ and $X \setminus V$ are disjoint $\mathcal{G}$-$g$-closed sets of a $\mathcal{G}$-$g$ -normal space $(X, \tau)$, by Theorem 3.9, there exist disjoint open sets $U$ and $W$ such that $\tau_{\mathcal{G}}\text{-cl}(F) \subseteq U$ and $\tau_{\mathcal{G}}\text{-cl}(X \setminus V) \subseteq W$. Now $U \subseteq X \setminus W \subseteq \tau_{\mathcal{G}}\text{-int}(V) \subseteq V$. Thus $F \subseteq \tau_{\mathcal{G}}\text{-cl}(F) \subseteq U \subseteq \tau_{\mathcal{G}}\text{-int}(V) \subseteq V$.

If we set $\mathcal{G} = [X]$ in the above theorem, we have the following result.

**Corollary 3.11.** Let $(X, \tau)$ be a $g$-normal space. If $F$ is a $g$-closed set and $V$ a $g$-open set containing $F$, then there exists an open set $U$ in $X$ such that $F \subseteq \mathrm{cl}(F) \subseteq U \subseteq \mathrm{int}(V) \subseteq V$.

**Definition 3.12.** Let $\mathcal{G}$ be a grill on a space $(X, \tau)$. Then $(X, \tau)$ is said to be $g$-$\mathcal{G}$normal if for each pair of disjoint closed sets $F$ and $K$, there exist disjoint $\mathcal{G}$-$g$-open sets $U$ and $V$ such that $F \subseteq U$ and $K \subseteq V$.

**Remark 3.13.** Since every open set is $\mathcal{G}$-$g$-open for any grill $\mathcal{G}$ on $X$, every normal space is $\mathcal{G}$-$g$-normal. That the converse is false is shown by the following example.

**Example 3.14.** Consider the grill $\mathcal{G} = \{\{b\}, \{a, b\}, \{b, c\}, X\}$ on the topological space $(X, \tau)$ where $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$. Then $(X, \tau)$ is clearly not normal. But

$(X, \tau)$ is $g$-$\mathcal{G}$-normal. In fact, $\Phi(\{a\}) = \phi$, $\Phi(\{a, b\}) = \{b\}$ and $\Phi(\{a, c\}) = \phi$. Thus every open set of $X$ is $\tau_{\mathcal{G}}$-closed and hence by Theorem 1.12, every subset of $X$ is $\mathcal{G}$-$g$-closed and so every subset of $X$ is $\mathcal{G}$-$g$-open.

**Theorem 3.15.** Let $\mathcal{G}$ be a grill on a space $(X, \tau)$. Then the following are equivalent:

($a$) $X$ is $g$-$\mathcal{G}$-normal.

($b$) For each closed set $F$ and each open set $V$ containing $F$, there exists a $\mathcal{G}$-$g$-open set $U$ such that $F \subseteq U \subseteq \tau_{\mathcal{G}}$-cl$(U) \subseteq V$.

**Proof.** ($a$) $\Rightarrow$ ($b$) : Let $F$ be a closed set and $V$ be an open set such that $F \subseteq V$. Then by ($a$), there exist disjoint $\mathcal{G}$-$g$-open sets $U$ and $W$ such that $F \subseteq U$ and $X \setminus V \subseteq W$. Now $U \cap W = \phi \Rightarrow U \cap \tau_{\mathcal{G}}$-int$(W) = \phi \Rightarrow \tau_{\mathcal{G}}$-cl$(U) \subseteq X \setminus \tau_{\mathcal{G}}$-int$(W)$. Since $X \setminus V \subseteq W$ where $W$ is $\mathcal{G}$-$g$-open, by Theorem 1.11, $X \setminus V \subseteq \tau_{\mathcal{G}}$-int$(W)$. Thus $F \subseteq U \subseteq \tau_{\mathcal{G}}$-cl$(U) \subseteq X \setminus \tau_{\mathcal{G}}$-int$(W) \subseteq V$.

($b$) $\Rightarrow$ ($a$) : Let $F$ and $K$ be any two disjoint closed subsets of $X$. Then by ($b$), there exists a $\mathcal{G}$-$g$-open set $U$ such that $F \subseteq U \subseteq \tau_{\mathcal{G}}$-cl$(U) \subseteq X \setminus K$. Put $V = X \setminus \tau_{\mathcal{G}}$-cl$(U)$. Then $U$ and $V$ are disjoint $\mathcal{G}$-$g$-open sets such that $F \subseteq U$ and $K \subseteq V$. Hence $(X, \tau)$ is $g$-$\mathcal{G}$-normal.

In [5], the following theorem was deduced:

**Theorem 3.16.** Let $\mathcal{G}$ be such a grill on a space $(X, \tau)$ that $PO(X)\setminus\{\phi\} \subseteq \mathcal{G}$. Then the following are equivalent:

(a) $X$ is normal.

(b) For each pair of disjoint closed sets $F$ and $K$, there exist disjoint $\mathcal{G}$-$g$-open sets $U$ and $V$ such that $F \subseteq U$ and $K \subseteq V$.

(c) For each closed set $F$ and any open set $V$ containing $F$, there is a $\mathcal{G}$-$g$-open set $U$ such that $F \subseteq U \subseteq \tau_{\mathcal{G}}$-cl$(U) \subseteq V$.

Now from Theorem 3.15 and Theorem 3.16, we get the following result :

**Corollary 3.17.** Let $\mathcal{G}$ be a grill on a space $(X, \tau)$ such that $PO(X) \setminus \{\phi\} \subseteq \mathcal{G}$. Then $(X, \tau)$ is $g$-$\mathcal{G}$-normal iff $(X, \tau)$ is normal.

**Theorem 3.18.** Let $\mathcal{G}$ be a grill on a space $(X, \tau)$. Let $F$ be closed and $K$ be $g$-closed in a

g-$\mathcal{G}$-normal space $(X, \tau)$ such that $F \cap K = \phi$. Then there exist disjoint $\mathcal{G}$-g-open sets $U$ and $V$ such that $\mathrm{cl}(K) \subseteq U$ and $F \subseteq V$.

**Proof.** Let $F$ be closed and $K$ be a g-closed set in a $\mathcal{G}$-g-normal space $X$ such that $F \cap K = \phi$. Since $K$ is g-closed, $\mathrm{cl}(K) \subseteq X \setminus F$, again since $X$ is g-$\mathcal{G}$-normal and $\mathrm{cl}(K) \cap F = \phi$, there exist disjoint $\mathcal{G}$-g-open sets $U$ and $V$ such that $\mathrm{cl}(K) \subseteq U$ and $F \subseteq V$.

## 4. GENERALIZED CONTINUOUS FUNCTIONS VIA GRILL

We begin this section by quoting the following definition from [1]:

**Definition 4.1.** A function $f: X \to Y$ is said to be generalized continuous (g-continuous, for short) if the inverse image of every closed set in $Y$ is g-closed in $X$.

In an analogous manner, we now define s follows:

**Definition 4.2.** A function $f: X \to Y$ is said to be generalized continuous with respect to some grill $\mathcal{G}$ on $X$ ($\mathcal{G}$-g-continuous, for short) if the inverse image of every closed set in $Y$ is $\mathcal{G}$-g-closed in $X$.

**Observation 4.3.** A function $f: X \to Y$ is $\mathcal{G}$-g continuous with respect to some grill $\mathcal{G}$ on $X$ iff the inverse image of every open set in $Y$ is $\mathcal{G}$-g-open in $X$.

**Remark 4.4.**

(i) It is shown in [1] that every continuous function is g-continuous and but not conversely.

(ii) It is now clear that every g-continuous function is $\mathcal{G}$-g-continuous; but the converse is false. In fact, let $X = \{a, b, c\}$, $\tau = \{\phi, \{c\}, \{b, c\}, X\}$ and $\mathcal{G} = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$; $Y = \{x, y\}$, $\sigma = \{\phi, \{y\}, Y\}$.

We define a function $f: (X, \tau, \mathcal{G}) \to (Y, \sigma)$ as follows:

$$f(a) = f(b) = y, \; f(c) = x.$$

Then $f$ is $\mathcal{G}$-g-continuous but it is not g-continuous. In fact, $F = \{x\}$ is closed in $(Y, \sigma)$, $f^{-1}(F) = \{c\}$ is $\mathcal{G}$-g-closed in $(X, \tau, \mathcal{G})$ but it is not g-closed.

**Definition 4.5.** A function $f: (X, \tau, \mathcal{G}_1) \to (Y, \sigma, \mathcal{G}_2)$ is said to be $(\tau\mathcal{G}_1, \tau\mathcal{G}_2)$-closed if the image of every $\tau\mathcal{G}_1$-closed set in $X$ is $\sigma\mathcal{G}_2$-closed in $Y$.

From now on, whenever $\mathcal{G}_1$(resp. $\mathcal{G}_2$) is a grill on a space $X$ (resp. on $Y$) and $A \subseteq X$ (resp. $\subseteq Y$) is generalized closed or open with respect to $\mathcal{G}_1$ (resp. $\mathcal{G}_2$), we shall write, to simplify notation, that $A$ is $\mathcal{G}$-g-closed or $\mathcal{G}$-g-open in $X$ (resp. $Y$).and hope that the context will make things clear.

**Theorem 4.6.** Let $A$ ($\subseteq X$) be a $\mathcal{G}$-g-closed set in a grill topological space $(X, \tau, \mathcal{G}_1)$. If $f$ : $(X, \tau, \mathcal{G}_1) \to (Y, \sigma, \mathcal{G}_2)$ is continuous and $(\tau\mathcal{G}_1, \sigma\mathcal{G}_2)$-closed, then $f(A)$ is a $\mathcal{G}$-g-closed set in $(Y, \sigma, \mathcal{G}_2)$.

**Proof.** Let $f(A) \subseteq U$ where $U$ is open in $Y$. Then $A \subseteq f^{-1}(U)$ and $f^{-1}(U)$ is open in $X$. Since $A$ is $\mathcal{G}$-g-closed, $\Phi(A) \subseteq f^{-1}(U)$ so that $A \cup \Phi(A) \subseteq f^{-1}(U)$. Thus $\tau\mathcal{G}_1$-cl$(A) \subseteq f^{-1}(U)$ and hence $f(\tau\mathcal{G}_1$-cl$(A) \subseteq U$. Since $f$ is $(\tau\mathcal{G}_1, \sigma\mathcal{G}_2)$-closed, $f(\tau\mathcal{G}_1$-cl$(A))$ is $\sigma\mathcal{G}_2$-closed and $\sigma\mathcal{G}_2$-cl$(f(A)) = f(\tau\mathcal{G}_1$-cl$(A) \subseteq U$.Hence $f(A)$ is $\mathcal{G}$-g-closed in $Y$.

However the image of a $\mathcal{G}$-g-closed set need not be $\mathcal{G}$-g-closed under continuous function as is shown below.

**Example 4.7.** Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\mathcal{G}_1 = \{\{c\}, \{a, c\}, \{b, c\}, X\}$; $Y = \{x, y, z\}$, $\sigma = \{\phi, \{x\}, \{y\}, \{x, y\}, Y\}$, $\mathcal{G}_2 = \{\{x\}, \{y\}, \{x, y\}, \{x, z\}, \{y, z\}, Y\}$.

We define a function $f : (X, \tau, \mathcal{G}_1) \to (Y, \sigma, \mathcal{G}_2)$ as follows :

$$f(a) = x,\ f(b) = f(c) = y.$$

Then $f$ is continuous. Now $A = \{a, c\}$ is $\mathcal{G}$-g-closed but its image $f(A) = \{x, y\}$ is not $\mathcal{G}$-g-closed.

**Definition 4.8.** A function $f : X \to Y$, where $X$, $Y$ are two grill topological spaces, is said to be irresolute with respect to some grill $\mathcal{G}$ on $X$ ($\mathcal{G}$-g-irresolute, for short) if the inverse image of every $\mathcal{G}$-g-open set in $Y$ is $\mathcal{G}$-g-open in $X$.

**Theorem 4.9.** Let $f : X \to Y$ be open, $\mathcal{G}$-g-irresolute and surjective. Then

(*a*) $X$ is $\mathcal{G}$-g-regular $\Rightarrow$ $Y$ is $\mathcal{G}$-g-regular.

(*b*) $X$ is $\mathcal{G}$-g-normal $\Rightarrow$ $Y$ is $\mathcal{G}$-g-normal.

**Proof.** (*a*) Let $F$ be $\mathcal{G}$-g-closed in $Y$ and $y \in Y \setminus F$. Since $f$ is $\mathcal{G}$-g-irresolute, $f^{-1}(F)$ is $\mathcal{G}$-g-

closed in $X$. Also $x \notin f^{-1}(F)$ where $x \in f^{-1}(y)$. Since $X$ is $\mathcal{g}$-$g$-regular, there exist disjoint open sets $U$ and $V$ *in* X such that $x \in U$ and $f^{-1}(F) \subseteq V$. Since $f$ is open and surjective, $f(U)$ and $f(V)$ are disjoint open sets in $Y$ such that $y \in f(U)$ and $F \subseteq f(V)$. Hence $Y$ is $\mathcal{g}$-$g$-regular.

(*b*) The proof is quite similar to that of (a) and hence is omitted.

**Theorem 4.10.** Let $f : (X, \tau, \mathcal{g}_1) \to (Y, \sigma, \mathcal{g}_2)$ be continuous and a $(\tau\mathcal{g}_1, \sigma\mathcal{g}_2)$-closed injective mapping.

(*a*) If $Y$ be $\mathcal{g}$-$g$-regular then $X$ is $\mathcal{g}$-$g$-regular.

(*b*) If $Y$ be $\mathcal{g}$-$g$-normal then $X$ is $\mathcal{g}$-$g$-normal.

**Proof.** (*a*) Let $F$ be $\mathcal{g}$-$g$-closed in $X$ and $x \in X \setminus F$. Since $f$ is continuous and $(\tau\mathcal{g}_1, \sigma\mathcal{g}_2)$-closed, by Theorem 4.6. $f(F)$ is $\mathcal{g}$-$g$-closed in $Y$ and also $f(x) \notin f(F)$. Again since $Y$ is $\mathcal{g}$-$g$-regular, there exist disjoint open sets $U$ and $V$ such that $f(x) \in U$ and $f(F) \subseteq V$. Thus $x \in f^{-1}(U)$ and $F \subseteq f^{-1}(V)$. Since $f$ is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are open in $X$ and hence $X$ is $\mathcal{g}$-$g$-regular.

The proof of (b) is quite similar and left.

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# SOME IDENTITIES INVOLVING SUMS OF LUCAS NUMBERS

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**ABSTRACT :** Several combinatorial identities involving Lucas numbers. Fibonacci numbers and exponents of two are derived by the mathematical induction. The notion of the Lucas matrix $\mathcal{L}_n^{(s)}$ of type $s$, which contains Lucas numbers along the main diagonal parallels, is introduced. Regular case $s = 0$ is contained in the results obtained in [19]. In the present paper the case $s = 1$ is investigated. The inverse of the Lucas matrix $\mathcal{L}_n^{(1)}$ is derived using previously defind identities. Factorization of the general Pascal matrix as well as a particular factorization of the block-Pascal matrix in terms of the matrix $\mathcal{L}_n^{(1)}$ is given. Additional combinatorial identities referring to the generalized Fibonacci numbers, binomial coefficients and special functions are derived as implications of these matrix correlations.

**Key words :** Matrix inverse; Fibonacci numbers; Lucas numbers; Fibonacci matrix: Pascal matrix; Lucas matrix; Toeplitz matrix.

**AMS Subj. Class. :** 05A10, 05A19, 15A09, 11B39, 11B68.

## 1. INTRODUCTION

The Fibonacci numbers $\{F_n\}_{n=0}^{\infty}$ are terms of the sequence satisfying initial conditions $F_0 = 0$, $F_1 = 1$, and the recurrence relation $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$. Close companions to the Fibonacci numbers, are the Lucas numbers $\{L_n\}_{n=0}^{\infty}$, which follow the same recursive pattern, but begin with $L_0 = 2$ and $L_1 = 1$. The Fibonacci and Lucas sequences has been discussed in many articles and books (sec. for example [9]). Fibonacci and Lucas numbers have long interested mathematicians for theoretical development and applications. For example, an application of these numbers in graph theory has been investigated in [21], while an interesting

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application of Fibonacci numbers in codding theory has been studied in [17]. The ratio of two consecutive of these numbers converges to the Golden section $\alpha = (1 + \sqrt{5})/2$, which application appears in many research areas, particularly in Physics. Engineering, Architecture, Nature and Art. Naschie and Marek-Crnjac gave some examples of the Golden ratio in Theoretical Physics and Physics of High Energy Particles [13, 14, 15]. A few mathematical concepts of similarity and proportion are known to be critical in understanding the growth processes in the natural world [23]. For example, the Fibonacci series is well known to lie at the heart of plant growth and living organisms [10]. The Lucas relationship within the 20 canonical amino acids has been shown in [23].

In the present paper we derive some additional combinatorial identities expressing sums which include Lucas numbers. Our results are derived using lower triangular Toeplitz matrices (Toeplitz matrices are constant along the diagonals) containing Lucas numbers.

The outline of this paper is as follows. The explicit representation of the inverse of the matrix $\mathcal{L}_n^{(1)}$ is derived in the second section using auxiliary combinatorial identity referring to Lucas numbers. A factorization of the generalized Pascal matrix in terms of the Lucas matrix $\mathcal{L}_n^{(1)}$ are considered in Section 3. Some combinatorial identities involving binomial coefficients, the Lucas numbers and special functions are derived as corollaries in the third section. In the fourth section we generalize principles used in the the papers [12, 19, 18, 26] as well as in the previous section to a more general class of Pascal-like matrices. For this purpose, we define generalized block-Pascal matrices of type $s$ as a generalization of generalized Pascal matrices. Factorization of the block-Pascal matrix of type 2 with respect to the matrix $\mathcal{L}_n^{(1)}$ are derived in accordance with the block structure of the inverse of the matrix $\mathcal{L}_n^{(1)}$. Additional combinatorial identities identities involving the Lucas numbers are derived by observing the particular case $x = 1/2$, $a = 2$, $b = 1$ we derived.

## 2. DEFINITIONS AND MOTIVATION

For the sake of completeness, we restate basic facts about the special functions needed in the present paper. The Euler gamma function is denoted by $\Gamma(n)$,

$$(a)_n = a(a+1) \ldots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$$

is well-known Pochhammer function,

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots b_q; \lambda) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \cdot \frac{\lambda^k}{k!}$$

is a generalized hypergeometric function, and

$${}_p\widetilde{F}_q[\{a_1, \ldots, a_p\}, \{b_1, \ldots, b_q\}, z] = {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) \,/\, (\Gamma(b_1) \ldots \Gamma(b_q))$$

defines the regularized generalized hypergeometric function.

Various types of lower triangular Toeplitz matrices which include various types of combinatorial numbers were investigated in [1, 2, 4, 6, 24, 25]. The generalized Pascal matrix of the first kind $\mathcal{P}_n[x] = [p_{i,j}[x]]$, $i, j = 1, \ldots, n$ is defined in [4]:

$$p_{i,j}[x] = \begin{cases} x^{i-j}\binom{i-1}{j-1} & i \geq j, \\ 0 & i < j. \end{cases} \tag{2.1}$$

In the case $x = 1$, the generalized Pascal matrix of the first kind reduces to frequently used Pascal matrix $\mathcal{P}_n = [p_{i,j}]$, $i, j = 1, \ldots, n$, which is defined in [3, 4].

The $n \times n$ Fibonacci matrix $\mathcal{F}_n = [f_{i,j}]$ $(i, j = 1, \ldots, n)$ is defined in [11], arranging the Fibonacci numbers on the main diagonal and below :

$$f_{i,j} = \begin{cases} F_{i-j+1}, & i-j+1 \geq 0, \\ 0, & i-j+1 < 0 \end{cases}. \tag{2.2}$$

The inverse and Cholesky factorization of the Fibonacci matrix are given in [11]. A first kind as well as the second kind factorization of the Pascal matrix in terms of the Fibonacci matrix are studied in [12]. Very helpful consequences of these matrix relations are various combinatorial identities involving the binomial coefficients and the Fibonacci numbers.

As an analogy of the Fibonacci matrix, the $n \times n$ Lucas matrix $\mathcal{L}_n = [l_{i,j}]$ $(i, j = 1, \ldots, n)$ is defined in [27]:

$$l_{i,j} = \begin{cases} L_{i-j+1}, & i-j \geq 0, \\ 0, & i-j < 0. \end{cases} \tag{2.3}$$

In the particular case $a = 2$, $b = 1$ the generalized Fibonacci matrix from [19] reduces to a generalization of the Lucas matrix.

**Definition 2.1** *The Lucas matrix of type s and of the order n, denoted by* $\mathcal{L}_n^{(s)} = \left[l_{i,j}^{(s)}\right]$, *i, j = 1, ..., n, is given by*

$$l_{i,j}^{(s)} = \begin{cases} L_{i-j+1}, & i-j+s \geq 0 \\ 0, & i-j+s < 0 \end{cases} \quad i,\ j = 1,\ \dots,\ n. \tag{2.4}$$

The only two cases that generate regular matrices of this type are $s = 0$ and $s = 1$ (for the proof see [19]). In this paper we consider the regular matrix $\mathcal{L}_n^{(1)}$, corresponding to the choice $s = 1$ in (2.4). The case $s = 0$ is investigated in [27]. In all other cases the matrix $\mathcal{L}_n^{(s)}$ is singular. These cases are not considered in [27].

In the study [22], some new properties of Lucas numbers with binomial coefficients were obtained to write Lucas sequences in a new direct way. Sums of squares of odd and even terms of the Lucas sequence and alternating sums of their products were investigated in [5]. In the paper [7], some known fibonacci and Lucas sums were derived by using proofs based on the usage of two appropriate matrices. A few representations of Lucas numbers in terms of the generalized hypergeometric functions were derived in [16]. Several combinatorial identities involving binomial coefficients, powers of two and the Lucas numbers are known. For example, we restate identities from [8]:

$$\sum_{k=0}^{n} \binom{n}{k} 2^k L_k = L_{3n}, \quad \sum_{k=0}^{n} \binom{n}{k} L_k = L_{2n}.$$

In the present paper we derive interesting representations for sums of the form $\Sigma 2^{k-1} L_k$ and $\Sigma k \cdot 2^{k-1} L_k$. Further, we derive various more general combinatorial identities involving Lucas numbers, binomial co-effcients and some special functions. These identities are derived continuing the general principles from [12, 20, 18, 26, 27] and using explicit representation of the inverse of the matrix $\mathcal{L}_n^{(1)}$ and the first kind factorization of the Pascal matrix.

## 3. SOME IDENTITIES OF SUMS INVOLVING LUCAS NUMBERS

The following combinatorial identity gives the exact value of the sum $\Sigma(n) = \sum_{k=0}^{n} 2^{k-1} L_k$.

**Lemma 3.1** *The following recurrence relation is valid for arbitrary integer $n \geq 0$.*

$$\Sigma(n) = \sum_{k=0}^{n} 2^{k-1} L_k = \frac{2^n}{5} (2L_{n+2} - L_{n+1}) \tag{3.1}$$

$$= 2^n F_{n+1}. \tag{3.2}$$

**Proof.** The identity (3.1) can be verified by the principle of the mathematical induction. The inductive step follows from

$$\sum_{k=0}^{n+1} 2^{k-1} L_k = 2^n L_{n+1} + \frac{2^n}{5} (2L_{n+2} - L_{n+1})$$

$$= \frac{2^n}{5} (2L_{n+2} + 4L_{n+1}) = \frac{2^{n+1}}{5} (2L_{n+3} - L_{n+2}).$$

Finally, using

$$2L_{n+2} - L_{n+1} = L_{n+2} - L_n = 5F_{n+1}$$

the identity (3.2) is verified.

**Corollary 3.1.** *For arbitrary integer $n \geq 0$, the next identities hold:*

$$\Sigma(n) = \frac{1}{40}\left(1-\sqrt{5}\right)^{-n-2}\left(-\left(\sqrt{5}-5\right)(-4)^{n+2} + \left(5+\sqrt{5}\right)\left(6-2\sqrt{5}\right)^{n+2}\right) \tag{3.3}$$

$$= \frac{\sqrt{5}}{10}\left(\left(1+\sqrt{5}\right)^{n+1} - \left(1-\sqrt{5}\right)^{n+1}\right). \tag{3.4}$$

**Proof.** The proof can be completed applying the analog of Binet's Fibonacci number formula for Lucas numbers:

$$L_k = \left(\frac{1-\sqrt{5}}{2}\right)^k + \left(\frac{1+\sqrt{5}}{2}\right)^k, \ k \geq 0. \tag{3.5}$$

to (3.1) we get.

**Corollary 3.2** *The following representation of Lucas numbers is valid for arbitrary integer* $n \geq 0$:

$$F_{n+1} = \frac{\sum\limits_{k=0}^{n} {}_2F_1\left(-\frac{1}{2}k, -\frac{1}{2}k+\frac{1}{2}; \frac{1}{2}; 5\right)}{2^n} \tag{3.6}$$

**Proof.** The proof can be derived applying the next representation of the Lucas numbers from [16]

$$L_k = \left(\frac{1}{2}\right)^{k-1} {}_2F_1\left(-\frac{1}{2}k, -\frac{1}{2}k+\frac{1}{2}; \frac{1}{2}; 5\right)$$

to (3.2).

**Proposition 3.1** (*a*) *The sum* $2\Sigma(n)$ *ends with* $\mathrm{mod}(2(n+1), 10)$.

(*b*) *The sum* $\Sigma(n)$ *ends with* $\mathrm{mod}(6n-4, 10)$.

**Proof.** (a) Since each term $2^k L_k$ ends with 2, for each integer $k$ (see [8]), the expression $2\Sigma(n)$ contains $n+1$ terms obeying this property.

In the following lemma we introduce an additional combinatorial identity involving the Lucas numbers.

**Lemma 3.2.** *Consider integers* $i$, $j$ *and* $s$, $p$ *which satisfy* $p \geq 3$, $s \leq p \leq i+1$. *The partial convolution defined by*

$$I_1(s, p, i, j) = \sum_{k=s}^{p} 2^{j-k} L_{i-k+1} = 2^{j-1} \sum_{k=s-1}^{p-1} 2^{-k} L_{i-k} \tag{3.7}$$

*satisfies the following equality :*

$$I_1(s, p, i, j) = \begin{cases} 2^{j-i}\left(\Sigma(i-s+1) - \Sigma(i-p)\right), & p \leq i, \\ 2^{j-i}\Sigma(i-s+1), & p = i+1. \end{cases} \tag{3.8}$$

**Proof.** The first case of the proof follows from

$$\sum_{k=s}^{p} 2^{j-k} L_{i-k+1} = 2^{j-i} \sum_{k=s}^{p} 2^{i-k} L_{i-k+1} = 2^{j-i} \sum_{j=i-p+1}^{i-s+1} 2^{j-1} L_j$$

$$= 2^{j-i} \left( \sum_{j=0}^{i-s+1} 2^{j-1} L_j - \sum_{j=0}^{i-p} 2^{j-1} L_j \right)$$

and (3.1). Since

$$I_1(s,\ i+1,\ i,\ j) = I_1(s,\ i,\ i,\ j) + 1 = \Sigma(i-s+1) - \Sigma(i-i) + 1 = \Sigma(i-s+1),$$

the second case is also verified.

Now, we find explicit representation of the sum

$$\sum_{k=0}^{n} k \cdot 2^{k-1} L_k .$$

**Theorem 3.1** *The following indentity is valid for arbitrary integer* $n \geq 1$ :

$$\Sigma_1(n) = \sum_{k=0}^{n} k \cdot 2^{k-1} L_k = \frac{1+2^n((n-1)L_{n+2}+(n+1)L_n)}{5} \tag{3.9}$$

$$= \frac{1+2^n((3n+5)L_n+(n+1)L_{n-1})}{5} - 2 \cdot \Sigma(n) . \tag{3.10}$$

**Proof.** The identity (3.9) is verified by the mathematical induction. The base of the induction is easy for the verification and the inductive step follows from

$$\Sigma_1(n) = (n+1)2^n L_{n+1} + \frac{1+2^n((n-1)L_{n+2}+(n+1)L_n)}{5}$$

$$= \frac{1+2^n((n-1)L_{n+2}+(5n+5)L_{n+1}+(n+1)L_n)}{5}$$

$$= \frac{1+2^n(2nL_{n+3}-(n+1)L_{n+2}+(3n+5)L_{n+1}+(n+1)L_n)}{5}$$

$$= \frac{1+2^n(2nL_{n+3}+(2n+4)L_{n+1})}{5}.$$

Identity (3.10) is a consequence of (3.9).

## 4. IDENTITIES BASED ON THE USAGE OF GENERALIZED PASCAL MATRIX

In this section we investigate correlations between the matrix $\mathcal{L}_n^{(1)}$ and the generalized Pascal matrices. Several new combinatorial identities are derived as implications.

**Theorem 4.1.** *The inverse* $\left(\mathcal{L}_n^{(1)}\right)^{-1} = [l'_{i,j}]$ $(i, j = 1, \ldots, n)$ *of the Lucas matrix* $\mathcal{L}_n^{(1)}$ *is defined by*

$$l'_{i,j} = \begin{cases} 3, & i>2,\ j\le n-2,\ i=j+1; \\ 5\cdot 2^{j-i}, & i>2,\ j\le n-2,\ i\le j; \\ (-1)^i\cdot 2^{j-i}, & i\le 2,\ j\le n-2,\ (i,j)\ne(2,1); \\ (-1)^{n-j+1}\cdot 2^{j-i}, & i>2,\ j>n-2,\ (i,j)\ne(n,\ n-1); \\ (-1)^{n-j+i-1}\cdot\frac{2^{j-i}}{5}, & i\le 2,\ j>n-2; \\ 0, & i>j+2; \\ 1, & \textit{otherwise}. \end{cases} \tag{4.1}$$

**Proof.** To simplify notation, let us denote $c_{i,j} = \sum_{k=1}^{n} l_{i,k}l'_{k,j}$. First, we want to prove $c_{i,i} = 1$, $i = 1, 2, \ldots, n$. Three different possibilities are feasible.

$(C_1)$ : It is easy to prove that $c_{1,1} = 1$. In the case $i = 2$ immediately follows

$$c_{2,2} = -2L_2 + L_1 + 3L_0 = 1.$$

$(C_2)$ : In the cae $2 < i \le n-2$, the value $c_{i,i}$ is equal to

$$c_{i,i} = \sum_{k=1}^{n} l_{i,k}l'_{k,i} = l_{i,1}l'_{1,i} + l_{i,2}l'_{2,i} + \sum_{k=3}^{i} l_{i,k}l'_{k,i} + l_{i,i+1}l'_{i+1,i}$$

$$= -2^{i-1}L_i + 2^{i-2}L_{i-1} + \sum_{k=3}^{i} 5\cdot 2^{i-k}L_{i-k+1} + 3L_0.$$

The result of Lemma 3.2 implies

$$\sum_{k=3}^{i} 5 \cdot 2^{i-k} L_{i-k+1} = 5I_1(3,\ i,\ i,\ i) = 5(\Sigma(i-2) - 1)$$

and later, taking into account (3.1), one can verify

$$c_{ij} = -\ 5\Sigma(i-2) + 5(\Sigma(i-2) - 1) + 3L_0 = 1.$$

$(C_3)$ : Finally, in the case $i = n$ (proof for $i = n - 1$ is very similar), it follows that

$$\begin{aligned} c_{n,\,n} &= \sum_{k=1}^{n} l_{n,\,k} l'_{k,\,n} = l_{n,\,1} l'_{1,\,n} + l_{n,\,2} l'_{2,\,n} + \sum_{k=3}^{n} l_{n,\,k} l'_{k,\,n} \\ &= \frac{2^{n-1}}{5} L_n - \frac{2^{n-2}}{5} L_{n-1} - \sum_{k=3}^{n} 2^{n-k} L_{n-k+1}. \end{aligned}$$

Applying the results of Lemma 3.1 and Lemma 3.2, this part of the proof can be simply completed.

Now, we want to prove that $c_{i,\,j} = 0$, for $i \neq j$. Similarly as in the first part of the proof, several different cases can be distinguished.

$(D_1)$ : It is casy to see that $c_{i,\,1} = 0$ in the case $i > 1$.

$(D_2)$ : In the case $1 < j < n - 1$, $i > j$ the following holds:

$$\begin{aligned} c_{i,\,j} &= \sum_{k=1}^{n} l_{i,\,k} l'_{k,\,j} = l_{i,\,1} l'_{1,\,j} + l_{i,\,2} l'_{2,\,j} + \sum_{k=3}^{j} l_{i,\,k} l'_{k,\,j} + l_{i,\,j+1} l'_{j+1,\,j} + l_{i,\,j+2} l'_{j+2,j} \\ &= -2^{j-1} L_i + 2^{j-2} L_{i-1} + 5 \sum_{k=3}^{j} L_{i-k+1} 2^{j-k} + 3L_{i-j} + L_{i-j-1}. \end{aligned}$$

Applying the result of Lemma 3.2 on the sum

$$\sum_{k=3}^{j} 2^{j-k} L_{i-k=1} = I_1(3,\ j,\ i,\ j),$$

one can verify

$$c_{i,\,j} = -\ 2^{j-1}\Sigma(i-2) + 5 \cdot 2^{j-i}(\Sigma(i-2) - \Sigma(i-j)) + 3L_{i-j} + L_{i-j-1}$$

$$= -5\cdot 2^{j-i}\cdot \frac{2^{i-j}}{5}(2L_{i-j+2} - L_{i-j+1}) + 2L_{i-j} + L_{i-j+1} = 0.$$

$(D_3)$ : In the case $1 < j < n - 1$, $i < j$, values $c_{i,j}$ are defined as

$$c_{i,j} = l_{i,1}l'_{1,j} + l_{i,2}l'_{2,j} + \sum_{k=3}^{i+1} l_{i,k}l'_{k,j} = -2^{j-1}L_i + 2^{j-2}L_{i-1} + 5\sum_{k=3}^{i+1} L_{i-k+1}2^{j-k}.$$

An application of the second case of (3.8) in conjunction with the results of Lemma 3.2, further implies :

$$c_{i,j} = -5\cdot 2^{j-i}(\Sigma(i-2) - \Sigma(i-2)) = 0.$$

$(D_4)$ : Finally, in the case $j = n$ (proof for $j = n - 1$ can be accomplished similarly), values $c_{i,n}$ are equal to

$$c_{i,n} = l_{i,1}l'_{1,n} + l_{i,2}l'_{2,n} + \sum_{k=3}^{i+1} l_{i,k}l'_{k,n} = \frac{1}{5}\left(2^{n-1}L_i - 2^{n-2}L_{i-1}\right) - \sum_{k=3}^{i+1} 2^{n-k}L_{i-k+1}.$$

Using the result from Lemma 3.2 together with the second case of (3.8) we have

$$5\sum_{k=3}^{i+1} 2^{n-k}L_{i-k+1} = I_1(3, i+1, i, n) = 2^{n-i}\Sigma(i-2),$$

which implies $c_{i,n} = 0$.

Hence, we prove $\mathcal{L}_n^{(1)}\left(\mathcal{L}_n^{(1)}\right)^{-1} = I_n$, where $I_n$ is the $n \times n$ identity matrix. In a similar way one can verify $\left(\mathcal{L}_n^{(1)}\right)^{-1}\mathcal{L}_n^{(1)} = I_n$ and the proof is completed.

According to Lemma 4.1, it is observable that $\left(\mathcal{L}_n^{(1)}\right)^{-1}$ possesses the following block matrix form :

$$\left(\mathcal{L}_n^{(1)}\right)^{-1} = \left[\begin{array}{c|c} \left(\mathcal{L}_n^{(1)}\right)^{-1}{}_{2|,(n-2)|} & \left(\mathcal{L}_n^{(1)}\right)^{-1}{}_{2|,|2} \\ \hline \left(\mathcal{L}_n^{(1)}\right)^{-1}{}_{|(n-2),(n-2)|} & \left(\mathcal{L}_n^{(1)}\right)^{-1}{}_{|(n-2),|2} \end{array}\right], \tag{4.2}$$

where the notations $k|$ (resp. $|k$) in the first indices denote the first (resp. last) $k$ rows and $k|$ (resp. $|k$) in the second indices denote the first (resp. last) $k$ columns.

**Example 4.1** *The* $6 \times 6$ *inverse Lucas matrix of type* 1 *is equal to*

$$\left(\mathcal{L}_6^{(1)}\right)^{-1} = \left[\begin{array}{cccc|cc} -1 & -2 & -4 & -8 & -\frac{16}{5} & -\frac{32}{5} \\ 1 & 1 & 2 & 4 & \frac{8}{5} & -\frac{16}{5} \\ \hline 1 & 3 & 5 & 10 & 4 & -8 \\ 0 & 1 & 3 & 5 & 2 & -4 \\ 0 & 0 & 1 & 3 & 1 & -2 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right].$$

It is interesting to point out that the blocks of the Lucas inverse matrix of type 1 are almost constant along diagonals, which means that the matrix $\left(\mathcal{L}_n^{(1)}\right)^{-1}$ is close to block Toeplitz matrix. Although (4.1) looks robust, it is not difficult to see that three blocks of the matrix $\left(\mathcal{L}_n^{(1)}\right)^{-1}$ are symmetric in relation to anti-diagonal and are almost Toeplitz (differs only in signs).

**Remark 4.1** *To store the matrix* $\left(\mathcal{L}_n^{(1)}\right)^{-1}$ *we only need to register in the memory: first row, element in position* $(2, n-1)$ *and first row of the* $(n-2) \times (n-2)$ *toeplitz matrix which is in the left-lower block. This means that we have to store only the vector with* $(2n-1)$ *elements, which is the same size for storing arbitrary Toeplitz matrix.*

A factorization of the Pascal matrix given in terms of the Lucas matrix of type 1 is presented in the following lemma.

**Lemma 4.1** *For arbitrary integer* $n > 4$, *the generalized Pascal matrix and the Lucas matrix of type* 1 *satisfy*

$$\mathcal{P}_n[1/2] = \mathcal{H}_n^{(1)}[1/2]\mathcal{L}_n^{(1)},$$

*where the matrix*

$$\mathcal{H}_n^{(1)}[1/2]=\left[h_{i,j}^{(1)}\left(\tfrac{1}{2}\right)\right]$$

*is defined by*

$$h_{i,j}^{(1)}\left(\tfrac{1}{2}\right)=\begin{cases}2^{j-i}\left(5\cdot 2^{i-1}-4i-2+6\binom{i-1}{j}+4\binom{i-1}{j+1}-5\binom{i-1}{j}{}_2F_1(1,j-i+1;j+1;-1)\right) & 1<j\le n-2;\\ 2^{j-1}(-1)^{n-j}\left(\frac{1+2i}{5}2^{-i+2}-1\right), & j>n-2, (i,j)\ne(n,n-1);\\ 2^{n-2}-\frac{3+n}{10}, & (i,j)=(n,n-1);\\ 2^{1-i}(1+2i(i-2)), & j=1.\end{cases} \tag{43}$$

**Proof.** It suffices to verify the matrix equality $\mathcal{H}_n^{(1)}[1/2] = \mathcal{P}_n[1/2]\left(\mathcal{L}_n^{(1)}\right)^{-1}$. Let us denote by $c_{ij} = \sum_{k=1}^{n} p_{i,k}l'_{k,j}$ and observe analogous cases as in (4.3).

$$\begin{aligned} c_{ij} &= \sum_{k=1}^{n} p_{i,k}l'_{k,j} = p_{i,1}l'_{1,j}+p_{i,2}l'_{2,j}+\sum_{k=3}^{j} p_{i,k}l'_{k,j}+p_{i,j+1}l'_{j+1,j}+p_{i,j+2}l'_{j+2,j}\\ &= -2^{j-i}+(i-1)2^{j-i}+5\sum_{k=3}^{j}\binom{i-1}{k-1}2^{j-i}+3\binom{i-1}{j}2^{j-i+1}+\binom{i-1}{j+1}2^{j-i+2}\\ &= 2^{j-i}\left[i-2+5\sum_{k=3}^{j}\binom{i-1}{k-1}+6\binom{i-1}{j}+4\binom{i-1}{j+1}\right]\end{aligned}$$

After the application of

$$\begin{aligned}\sum_{k=3}^{j}\binom{i-1}{k-1} &= 2^{i-1}-i-\frac{\Gamma(i)\,{}_2\widetilde{F}_1[\{1,-i+j+1\}\cdot\{j+1\},-1]}{\Gamma(i-j)}\\ &= 2^{i-1}-i-\binom{i-1}{j}\,{}_2F_1(1,j-i+1,j+1,-1),\end{aligned}$$

the first case can be verified.

For $j > n-2$, $(i, j) \ne (n, n-1)$, we have

$$c_{i,j} = \sum_{k=1}^{n} p_{i,k}l'_{k,j} = p_{i,1}l'_{1,j}+p_{i,2}l'_{2,j}+\sum_{k=3}^{i} p_{i,k}l'_{k,j}$$

$$= \left(\frac{1}{2}\right)^{i-1}\frac{(-1)^{n-j}2^{j-1}}{5} + (i-1)\left(\frac{1}{2}\right)^{i-2}\frac{(-1)^{n-j+1}2^{j-2}}{5} + (-1)^{n-j+1}\sum_{k=3}^{i}\binom{i-1}{k-1}\left(\frac{1}{2}\right)^{i-k}2^{j-k}$$

$$= 2^{j-i}(-1)^{n-j}\left[\frac{1}{5} - \frac{i-1}{5} - \sum_{k=3}^{i}\binom{i-1}{k-1}\right].$$

The approval follows straight having in mind the identity

$$\sum_{k=3}^{i}\binom{i-1}{k-1} = 2^{i-1} - i.$$

In the case $(i, j) = (n, n-1)$, we have

$$c_{n,\,n-1} = \sum_{k=1}^{n} p_{n,k}r_{k,n-1} = p_{n,1}r_{1,n-1} + p_{n,2}r_{2,n-1} + \sum_{k=3}^{n-1} p_{n,k}r_{k,n-1} + 1$$

$$= -\left(\frac{1}{2}\right)^{n-1}\frac{2^{n-2}}{5} + (n-1)\left(\frac{1}{2}\right)^{n-2}\frac{2^{n-3}}{5} + \sum_{k=3}^{n-1}\binom{n-1}{k-1}\left(\frac{1}{2}\right)^{n-k}2^{n-k-1} + 1$$

$$= -\frac{2}{5} + \frac{2}{5}(n-1) + \frac{1}{2}\sum_{k=3}^{n-1}\binom{n-1}{k-1} + 1.$$

After applying

$$\sum_{k=3}^{n-1}\binom{n-1}{k-1} = 2^{n-1} - n - 1,\ n > 4,$$

it is possible to obtain

$$c_{i,\,j} = 2^{n-2} - \frac{3+n}{10}.$$

It is obvious that for $j = 1$ the following is valid

$$c_{i,1} = 2^{3-i}\left(-\frac{1}{4} + \frac{1}{2}\binom{i-1}{1} + \binom{i-1}{2}\right),$$

which immediately produces the last case in (4.3). We get that $c_{i,j} = h_{i,j}^{(1)}(1/2)$ for $i, j = 1, \ldots, n$ and the proof is completed.

The next identity follows from Lemma 4.1.

**Theorem 4.2** *The following recurrence relation is valid for arbitrary integers* $n \geq 2$, $1 \leq i \leq n-1$ :

$$2i(i-2)+\sum_{j=2}^{n-2} 2^{j-1}\left(5\cdot 2^{i-1}-4i-2+6\binom{i-1}{j}+4\binom{i-1}{j+1}-5\binom{i-1}{j}\,{}_2F_1(1,j-i+1;j+1;-1)\right)L_j$$

$$+\sum_{j=n-1}^{n} 2^{j-1}(-1)^{n-j}\left(\frac{2-i}{5}-\sum_{k=3}^{i}\binom{i-1}{k-1}\right)L_j=0. \tag{4.4}$$

**Proof.** The following notation is useful:

$$p_{i,j} = p_{i,j}[1/2],\ h_{i,j} = h_{i,j}^{(1)}(1/2).$$

The next identity could be derived from Lemma 4.1, taking into account (4.3) :

$$\left(\frac{1}{2}\right)^{i-1} = p_{i,1}=\sum_{j=1}^{n} h_{i,j}l_{j,1}=h_{i,1}l_{1,1}+\sum_{j=2}^{n-2} h_{i,j}l_{j,1}+\sum_{j=n-1}^{n} h_{i,j}l_{j,1}$$

$$= 2^{1-i}(1 + 2i(i-2)) +$$

$$+\sum_{j=2}^{n-2} 2^{j-i}\left(5\cdot 2^{i-1}-4i-2+6\binom{i-1}{j}+4\binom{i-1}{j+1}-5\binom{i-1}{j}\,{}_2F_1(1,j-i+1;j+1;-1)\right)L_j$$

$$+\sum_{j=n-1}^{n} 2^{j-i}(-1)^{n-j}\left(\frac{2-i}{5}-\sum_{k=3}^{n-1}\binom{i-1}{k-1}\right)L_j.$$

In the case $1 \leq i \leq n-1$, (4.4) immediately follows.

**Remark 4.2** (i) *In the case* $i = 1$, *the identity* (4.4) *reduces to* (3.1).

(ii) *In the case* $i = 2$, *the identity* (4.4) *becomes the trivial identity* $0 = 0$.

## 5. CONCLUSION

Several combinatorial identities involving Lucas numbers, Fibonacci numbers and exponents of two are derived by the mathematical induction. The starting point of our investigation is the notion of the Lucas matrix $\mathcal{L}_n^{(s)}$ of type $s$, which is constant along the diagonals and filled with the generalized Fibonacci numbers. This matrix introduced in [19]. The regular case $s = 0$ includes the results obtained in [19]. In the present paper, the case $s = 1$ is investigated. The inverse of the Lucas matrix $\mathcal{L}_n^{(1)}$ is generated using identities previously derived by the induction.

The principles used in the papers [12, 19, 18, 26] are applied in sections 3 and 4, following the specifies of these particular kinds of Toeplitz matrices. Using the identity for the sum involving generalized Fibonacci numbers, derived in Lemma 3.2, the explicit representation for the inverse matrix of $\mathcal{L}_n^{(1)}$ is given in Lemma 4.1. A first kind factorization of the generalized Pascal matrix in terms of the Lucas matrix $\mathcal{L}_n^{(1)}$ is derived in Theorem 4.1. Based upon these matrix correlations, several interesting additional combinatorial identities involving the Lucas numbers and the binomial coefficients are derived, observing the special case $x = 1/2$, $a = 2$, $b = 1$.

Remaining singular cases of these matrices ($s \neq 0, 1$), could be the interest of our future research.

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